SPECTRAL ANALYSIS AND LIMIT BEHAVIOURS IN A SPRING-MASS SYSTEM.

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Abstract. We consider a model for a damped spring-mass system that is a strongly damped wave equation with dynamic boundary conditions. In a previous paper we showed that for some values of the parameters of the model, the large time behaviour of the solutions is the same as for a classical spring-mass damper ODE. Here we use spectral analysis to show that for other values of the parameters, still of physical relevance and related to the effect of the spring inner viscosity, the limit behaviours are very different from that classical ODE.

1. **Introduction.** This paper deals with the large time behaviour of the solutions of the following problem for the strongly damped wave equation:

$$\begin{cases} u_{tt} - u_{xx} - \alpha\,u_{txx} = 0, & 0 < x < 1, \quad t > 0 \\ u(0,t) = 0 \\ u_{tt}(1,t) = -\varepsilon\,[u_x(1,t) + \alpha\,u_{tx}(1,t) + r\,u_t(1,t)] \end{cases} \tag{1}$$

This is a continuous model for a spring-mass-damper system, when possible differences in the internal deformation of the spring are considered. We also consider the stabilizing effects of the internal viscosity of the spring and of an external damper located at one of the ends of the system. In this equation, $u(x,t)$ stands for the longitudinal displacement at time t of the x particle of the spring, and the internal viscosity or damping (of the Kelvin-Voigt type) is represented in the equation by the parameter $\alpha \geq 0$. We also have an external damping, represented by $r > 0$ at the boundary conditions, which only acts onto the spring through the mass at the end $x = 1$. The amount of mass appears at the boundary conditions in (1) essentially as $1/\varepsilon$, with $\varepsilon > 0$. A more detailed explanation of the physical meaning of each term can be found in our previous works [14] and [15].

This problem with $\varepsilon = \alpha = 1$ and $r = 0$ was considered by M. Grobbelaar-van Dalsen in [5], where the author gave an appropriate functional setting to study the problem. Our functional framework is actually strongly inspired by that paper.

Also in that work, and using the techniques of the previous paper by P. Massat [12], the well posedness of the problem was proved. These results can also be read as that it generates an analytic semigroup.

The study of the semigroup generated by a wave equation with localized Kelvin-Voigt damping was also done by S. Chen, K. Liu and Z. Liu in [2] and in K. Liu and Z. Liu in [10]. Depending on the localization of this inner damping, the resulting semigroup could be either analytic or only of class $\mathcal{C}^0$. In that paper, the authors also discussed the exponential decay or not of solutions depending on some different types of damping. Although this will not be the main point of view of our work, the results obtained in the present paper could also be related to those decay estimates.

Before these works, an elastic beam equation with strong damping appeared as an example in the work [18] by D. L. Russell. In that paper only the case $\varepsilon = 0$ was considered. But, in spite of being a very particular problem, it is of interest as it contains the main characteristics of the case $\varepsilon > 0$ and small. In [18] a spectral analysis of the operator was done. This was also one of the main tools in our previous work [15], where a careful study of the spectrum of the operator involved in (1) revealed that for small positive ε the spectrum is essentially a perturbation of the explicit spectrum for $\varepsilon = 0$.

In the present paper an analysis of the spectrum is also used to study the large time behaviour of our problem. As we have seen, spectral analysis has also been applied to the study of different questions appearing in wave equations. For instance, P. Freitas in [3] used it to study stability in a semilinear wave equation with strong damping and homogeneous Dirichlet boundary conditions. The description of the spectrum that we obtain in the present paper coincides in some points with the results of the analysis done in [3].

The motion of a mass in a spring-mass-damper system is usually modelled by the second order ordinary differential equation of the damped oscillations, namely:

$$m\,u''(t) = -k\,u(t) - d\,u'(t). \tag{2}$$

where $k > 0$ is the recovery constant of the spring and $d \geq 0$ stands for the dissipation coefficient. In the model (2), the spring-mass system is treated from a space-homogeneous point of view, while in the model (1) the space-dependence of the spring particles is taken into account. Since we have considered in (1) a different model for the spring-mass-damper system it is reasonable to wonder if both approaches can be compared. Actually, our main question is if for large times the solutions of the partial differential equation model (1) tend to solutions of an ordinary differential equation of the same type as (2). In other words we want to study the existence or not of a limit ODE for the PDE model, at least for some values of the parameters. We understand this limit in the sense given by dominant eigenvalues, that is explained as follows.

Let us write problem (1) in the evolution form

$$\frac{d}{dt}V = A_\alpha V \quad (\alpha \geq 0) \tag{3}$$

whose appropriate functional setting will be presented in section 2 below. If we denote by $\sigma(A_\alpha)$ the spectrum of the operator A_α, then we say that a finite subset of isolated eigenvalues $\{\lambda_1, \ldots \lambda_k\} \in \sigma(A_\alpha)$ with finite algebraic multiplicities is *dominant* if there exist $\omega_1, \omega_2 \in \mathbb{R}$ such that:

$$\operatorname{Re}\lambda < \omega_2 < \omega_1 < \operatorname{Re}\lambda_i \ \ \forall i = 1, \ldots, k \ , \ \forall \lambda \in \sigma(A_\alpha) \setminus \{\lambda_1, \ldots, \lambda_k\}.$$

In this case, at least for sectorial operators, it is reasonable to say that the ordinary differential equation generated by $\{\lambda_1, \ldots \lambda_k\}$ is the *limit* equation for the problem (3) (see [15] for details). Also, a set $\{\lambda_1, \ldots \lambda_l\}$ of dominant eigenvalues is called *the maximal dominant* if $\operatorname{Re} \lambda_1 = \ldots = \operatorname{Re} \lambda_l$.

In the previous work [15] we proved that for fixed α, $r > 0$ and ε small enough, the partial differential equation model (1) admits two dominant eigenvalues. Therefore it was proved the existence of a second order ODE of the same type as (2) as the limit of our model when $t \to \infty$ and ε is sufficiently small. In a future work a nonlinear version of this limit $\varepsilon \to 0$ will also be studied.

However, the existence of such a limit ODE may not always occur. In the present paper, in contrast with [15], we will show three interesting situations for (1), all of them related with internal viscosity $\alpha \geq 0$, where either the nonexistence of a finite subset of dominant eigenvalues can be proved or where there exists such a finite subset but the resulting ODE is not of the same type as (2). Thus, the existence of a limit ODE of type (2) is not an automatic property for model (1) as in principle one may think, but it only holds in some regions of the space of parameters (ε, α, r). These three cases are summarized in the following statements, and will be developed in the sections below.

The first of these three cases is $\alpha = 0$, that is the purely elastic spring (with an external damper but without internal viscosity). This case will be treated in section 3. Although it may seem the most similar situation to the one modelled by the classical equation (2), we will show the nonexistence of a limit ODE in this case.

Theorem 1.1. *When $\alpha = 0$ we have the following results:*

(i) *The spectrum of A_α with $\alpha = 0$ consists only of eigenvalues $\{\lambda_n\}$, $n \in \mathbb{N}$, with strictly negative real part that approach the imaginary axis as $|\lambda_n| \to \infty$. Therefore, it does not exist a finite subset of dominant eigenvalues.*

(ii) *In this case, it can also be seen that all solutions tend to zero as $t \to \infty$, but there exist solutions which tend to zero with arbitrarily slow rate.*

The slow decay to zero of solutions is a phenomenon that has been observed in several other similar problems. It has been studied mainly from the point of view of Control Theory. In remark 1 in section 3 below we give some references to these works.

The second case is $\alpha \sim 0$, that is a spring with small internal viscosity, and will be considered in section 4. This case will be studied as a singular perturbation of the previous one, but will exhibit very different properties. In section 4 we will prove the following result:

Theorem 1.2. *A_α admits a finite subset of maximal dominant eigenvalues for each $\alpha > 0$ if α is small enough. But this set does not depend continuously on α as $\alpha \to 0$. More precisely, neither the number of these eigenvalues nor their positions are continuous on α as $\alpha \to 0$.*

Finally in section 5 we will study a case with large α, a spring with large internal damping. As we will prove, the asymptotic dynamics of this situation is also not well approximated by any ODE of type (2). And, as we also will see, a kind of infinite-dimensional overdamping will occur:

Theorem 1.3. *For some values of $\varepsilon > 0$ and $r > 0$, and for α large enough, the operator A_α does not admit a finite subset of dominant eigenvalues. Actually, all*

the eigenvalues $\{\lambda_n\}$, $n \in \mathbb{N}$, are real with $-\infty < \lambda_n < -1/\alpha$, with a subsequence λ_n^+ that accumulates at $-1/\alpha$ and the rest of them accumulating at $-\infty$.

These three results allow us to say that an ordinary differential equation of the type (2) may not be necessarily the most appropriate model to describe the dynamics of a viscous spring-mass-damper system. This is true at least in the three cases that we analyze, and we point out that these three cases have relevant physical meaning.

2. **Function spaces and operators.** As in many problems with dynamic boundary conditions it is appropriate to work in spaces whose elements are pairs of a function and its boundary value. Moreover, since we are writing a second order evolution equation as a first order system our phase spaces will consist of pairs of such pairs. In this point we are strongly influenced by the work of M. Grobbelaar-van Dalsen [5]. Part of the definitions and results presented in this section also appear in [15], but we summarize them here for a better comprehension of the subsequent results.

We define the following spaces:

$$X_0 = L^2(0,1) \times \mathbb{C}$$

$$X_1 = \{(u,\gamma) \in H^1(0,1) \times \mathbb{C}, \ u(0) = 0, \ u(1) = \gamma\} \subset H^1(0,1) \times \mathbb{C}$$

$$X_2 = \{(u,\gamma) \in H^2(0,1) \times \mathbb{C}, \ u(0) = 0, \ u(1) = \gamma\} \subset H^2(0,1) \times \mathbb{C}$$

and $\mathcal{H} = X_1 \times X_0$, that is a Hilbert space with the following equivalent inner product:

$$
\begin{aligned}
&\left\langle \begin{pmatrix} (u_1, u_1(1)) \\ (u_0, \gamma_0) \end{pmatrix}, \begin{pmatrix} (v_1, v_1(1)) \\ (v_0, \beta_0) \end{pmatrix} \right\rangle_{\mathcal{H}} \\
&= \left\langle (u_1, u_1(1)), (v_1, v_1(1)) \right\rangle_{X_1} + \left\langle (u_0, \gamma_0), (v_0, \beta_0) \right\rangle_{X_0} \\
&= \int_0^1 (u_1)_x \, (\overline{v}_1)_x \, dx \ + \ \int_0^1 u_0 \, \overline{v}_0 \, dx \ + \ \frac{1}{\varepsilon} \gamma_0 \, \overline{\beta}_0.
\end{aligned}
\tag{4}
$$

The square of the norm defined by this scalar product (4) can be seen as the total physical energy of the system: the first term as the elastic potential energy of the spring, the second as the spring kinetic energy and the third as the kinetic energy of the mass at the end. This norm in $\mathcal{H}$ will be denoted simply by $\| \cdot \|$.

We define $(A_\alpha, \mathcal{D}(A_\alpha))$ as follows:

$$
\mathcal{D}(A_\alpha) = \left\{ \begin{pmatrix} (u, u(1)) \\ (v, v(1)) \end{pmatrix} \in X_1 \times X_1, \ (u + \alpha \, v) \in H^2(0,1) \right\} \subset \mathcal{H}
$$

is the domain of A_α, which is:

$$
A_\alpha \begin{pmatrix} (u, u(1)) \\ (v, v(1)) \end{pmatrix} = \begin{pmatrix} (v, v(1)) \\ ((u + \alpha \, v)_{xx}, \ -\varepsilon \, (u + \alpha \, v)_x(1) \ - \ \varepsilon \, r \, v(1)) \end{pmatrix}.
$$

Then, equation (1) can be written as the evolution equation:

$$
\frac{d}{dt} V = A_\alpha V, \qquad t \in (0, \infty)
$$

with $V = \begin{pmatrix} (u, u(1)) \\ (u_t, u_t(1)) \end{pmatrix} \in \mathcal{D}(A_\alpha)$.

The previous definitions include the case $\alpha = 0$, but observe that in this case one has $\mathcal{D}(A_0) = X_2 \times X_1$. Now one has the following theorem on existence and uniqueness of solutions.

Theorem 2.1. *Consider A_α, the corresponding operator when $\alpha \geq 0$. Then:*

 (i) *The operator $(A_\alpha, \mathcal{D}(A_\alpha))$ with $\alpha > 0$ is the generator of an analytic semigroup in $\mathcal{H}$.*

 (ii) *The operator $(A_0, \mathcal{D}(A_0))$ is the generator of a $\mathcal{C}^0$ semigroup of contractions in $\mathcal{H}$.*

The proof of part (i) can be found in [5] in the context of B-evolutions, and also in [2] by different methods. The proof of part (ii) can be seen in [14], where it is used the fact that A_0 is a dissipative operator in $\mathcal{H}$ (in the sense of Lumer-Phillips theorem) with the particular energy or scalar product defined above in (4). It can also be seen that A_α with $\alpha > 0$ is dissipative with the same energy, although this is not used in the proof of (i). The dissipatedness of A_0 ensures that this operator generates a $\mathcal{C}^0$-contractive semigroup, and the results on the spectrum of this operator given in section 3 will make clear that the semigroup generated by A_0 cannot be analytic.

The main part of the results presented in the following sections will use a careful study of the spectra of the operators A_α. We will start with $\alpha = 0$ in section 3, where we will use the characteristic equation corresponding with problem (1) for $\alpha = 0$ and the completeness of the set of generalized eigenfunctions. In section 4 we will consider the perturbed case $\alpha \sim 0$, in which the analysis of the spectrum will strongly use the concept of generalized convergence. This is the convenient notion of convergence between operators in order to compare their spectra and so to discuss the existence of dominant eigenvalues. The analysis of the spectrum in section 5 will mainly rely on the analysis of the corresponding characteristic equation, but from a different point of view from the one used in section 3.

3. **The purely elastic spring or $\alpha = 0$.** The following lemma contains the main spectral properties of A_0. These will be used in the proof of Theorem 1.1.

Lemma 3.1.

 (i) *The set of generalized eigenfunctions of A_0 is complete in $\mathcal{H}$.*

 (ii) *The spectrum of A_0, $\sigma(A_0)$, consists only of isolated eigenvalues with finite algebraic multiplicity.*

 (iii) *There exists $C = C(\varepsilon, r) > 0$ such that $-C < Re\,\lambda < 0$ for all $\lambda \in \sigma(A_0)$.*

Proof.   (i) Remember that V is a generalized eigenfunction of A_0 corresponding to some eigenvalue λ if $(A_0 - \lambda I)^k\, V = 0$ for some $k \in \mathbb{N}$.

This property will be proved for the operator iA_0 instead of A_0. The following decomposition can be done:

$$(iA_0)\,V = TV + KV$$

for $V = \begin{pmatrix} (u, u(1)) \\ (v, v(1)) \end{pmatrix} \in \mathcal{D}(A_0) = X_2 \times X_1$ and

$$TV = \begin{pmatrix} (iv, iv(1)) \\ (iu_{xx}, -i\varepsilon\, u_x(1)) \end{pmatrix}, \quad KV = \begin{pmatrix} (0, 0) \\ (0, -i\varepsilon\, r\, v(1)) \end{pmatrix}$$

It is easy to see that T is a selfadjoint operator in $\mathcal{H}$ with the inner product (4), and that K is a continuous operator with finite rank. Also, the spectrum

of T is discrete, since it has a compact inverse. Then, by a general result of Gohberg and Kreĭn (see [4], Chp.V.10) the set of generalized eigenfunctions of $T + K$ is complete in $\mathcal{H}$.

(ii) This is an immediate consequence of the compactness of $A_0^{-1} \in \mathcal{L}(\mathcal{H}, \mathcal{H})$.

(iii) Consider the energy:

$$E(u(x,t)) = \frac{1}{2} \int_0^1 |u_x|^2 \, dx \, + \, \frac{1}{2} \int_0^1 |u_t|^2 \, dx \, + \, \frac{1}{2\varepsilon} |u_t(1)|^2$$

defined by (4). It can be seen that the following inequality holds for any $u(x,t)$ solution of problem (1) with $\alpha = 0$:

$$\frac{dE\,(u(x,t))}{dt} \geq -2\varepsilon r E(u(x,t))$$

which, by considering again solutions of the type $u(x,t) = e^{\lambda t}\,\varphi(x)$, immediately gives us the desired relation,

$$-C \leq \operatorname{Re}\lambda \quad \forall\, \lambda \in \sigma(A_0).$$

for any constant $C > \varepsilon r > 0$.

$\square$

Proof of theorem 1.1. The details of this proof can be found in [14], but we summarize here its main parts.

(i) If we consider solutions of (1) when $\alpha = 0$ of the form $u(x,t) = e^{\lambda t}\varphi(x)$, it can be easily seen by using the equation and the boundary conditions that

$$\varphi(x) = e^{\sqrt{\frac{\lambda^2}{1 - \lambda\alpha}}x} - e^{-\sqrt{\frac{\lambda^2}{1 - \lambda\alpha}}x}$$

and that the eigenvalues λ of A_0 are the roots of the following characteristic equation:

$$e^{2\lambda} = \frac{\lambda + \varepsilon r - \varepsilon}{\lambda + \varepsilon r + \varepsilon}. \tag{5}$$

By points (i) and (ii) of lemma 3.1, this set of eigenvalues is infinite and countable. Expressing each eigenvalue λ_n as $\lambda_n = a_n + ib_n$, $a_n, b_n \in \mathbb{R}$, we obtain from (5) the following system:

$$\begin{cases} e^{2a_n} \cos(2b_n) = \dfrac{(a_n + \varepsilon r)^2 - \varepsilon^2 + b_n^2}{(a_n + \varepsilon r + \varepsilon)^2 + b_n^2} \\[2mm] e^{2a_n} \sin(2b_n) = \dfrac{2\varepsilon b_n}{(a_n + \varepsilon r + \varepsilon)^2 + b_n^2} \end{cases} \tag{6}$$

From the results of lemma 3.1 and using the previous system (6), we see that if for a subsequence λ_k one has $\lim_{n_k \to \infty} a_{n_k} = a$ then, necessarily, $\lim_{n_k \to \infty} b_{n_k} = \infty$ and $a = 0$. Recalling also from lemma 3.1 that there exists $C > 0$ such that $\operatorname{Re}\lambda \in (-C, 0)$ for all $\lambda \in \sigma(A_0)$ one deduces that $\lim_{n \to \infty} a_n = 0$ for the whole sequence of eigenvalues.

As we have $\lim_{n \to \infty} \operatorname{Re}\lambda_n = 0$ but $\operatorname{Re}\lambda_n < 0$ for all $\lambda_n \in \sigma(A_0)$, we can conclude the nonexistence of a finite number of dominant eigenvalues when $\alpha = 0$ (see figure 1).

(ii) We have just seen that the asymptotic behaviour of solutions of model (1) with $\alpha = 0$ cannot be approached by an ordinary differential equation. Nevertheless, we are able to say something more about this asymptotic behaviour of the solutions.
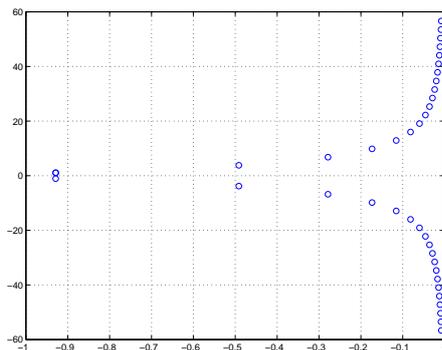
FIGURE 1. Spectrum of A_0.

From the fact that $\sigma(A_0) \subset \{\operatorname{Re}\lambda < 0\}$, that A_0 generates a semigroup of contractions (see theorem 2.1, part (ii)) and that the finite combinations of generalized eigenfunctions are dense in $\mathcal{H}$ (part (i) of lemma 3.1), one can easily deduce that all the solutions tend to zero as $t \to \infty$. But, in our case, it can also be proved that there exist solutions which tend to zero as slowly as we wish. This can be formulated in a more precise way: for every continuous function $\phi(t) : [0,\infty) \longrightarrow (0,\infty)$ with $\lim\limits_{t\to\infty} \phi(t) = 0$, there exists $V_0 \in \mathcal{H}$ such that

$$\limsup_{t\to\infty} \frac{\|e^{A_0 t}V_0\|_{\mathcal{H}}}{|\phi(t)|} = \infty.$$

The proof of this follows directly from the fact that A_0 is the generator of a $\mathcal{C}^0$ semigroup of contractions in $\mathcal{H}$ (see theorem 2.1) and the uniform boundedness principle.

The idea of these results can also be thought in terms of the eigenvalues of A_0, as all of them have strictly negative real part but they approach the imaginary axis as much as we wish.

$\square$

Remark 1. Some decaying rate results for other dissipative wave equations can be found in [7] or [9], for instance. In these works the responsible for dissipatedness is not an external damping but a coupling with a parabolic part. Other examples are that of [16], where the damping comes from an imposed external control at the boundary, or that of [13] that considers elastic solids with voids. In most of these papers, and also in the general results of [11], a polynomial decay rate for smoother solutions is proved, a result that perhaps is also true for the problem we are dealing with in this section.

Remark 2. As we said in the previous section, part (i) from Theorem 1.1 clearly shows that A_0 cannot generate an analytic semigroup.

4. **The small dissipation case or $\alpha \sim 0$.** We start with a description of $\sigma(A_\alpha)$ when $\alpha > 0$, which will also be used in section 5 for large values of α.

Theorem 4.1. *Suppose $\alpha > 0$, then:*

(i) *The point $\lambda = -1/\alpha$ is the only point in the essential spectrum of A_α (and it is not an eigenvalue). The rest of $\sigma(A_\alpha)$ consists only of infinitely many isolated eigenvalues with finite algebraic multiplicities. The eigenvalues are the roots of the following characteristic equation:*

$$e^{\frac{2\lambda}{\sqrt{1+\lambda\alpha}}} = \frac{\lambda - \varepsilon\sqrt{1+\lambda\alpha} + \varepsilon\,r}{\lambda + \varepsilon\sqrt{1+\lambda\alpha} + \varepsilon\,r} \qquad (7)$$

(ii) *If $\varepsilon r \alpha \geq 1$ then*

$$\sigma(A_\alpha) \subset \left(-\infty, \frac{-1}{\alpha}\right] \cup \left\{\left|\lambda + \frac{1}{\alpha}\right| < \frac{1}{\alpha}\right\} \quad \text{(see figure 2, left figure)}.$$

If $\varepsilon r \alpha < 1$ then:

$$\sigma(A_\alpha) \subset \left(-\infty, \frac{-1}{\alpha}\right] \cup \left\{\frac{1}{\alpha} - \varepsilon r < \left|\lambda + \frac{1}{\alpha}\right| < \frac{1}{\alpha}\right\} \quad \text{(see figure 2, right figure)}.$$
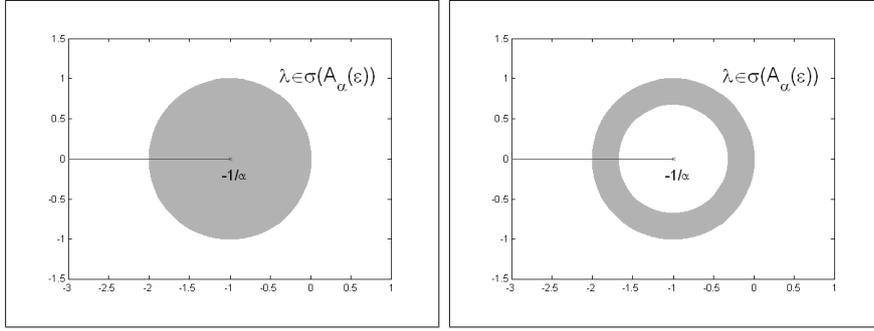

FIGURE 2. Localization of eigenvalues of $A_\alpha(\varepsilon)$ when $\varepsilon r \alpha \geq 1$ (left figure) and when $\varepsilon r \alpha < 1$ (right figure).

(iii) *The set of eigenvalues can be expressed as the union of two sequences, one that accumulates to $-1/\alpha$ and the other tending to $-\infty$.*

Remark 3. We use here the notion of essential spectrum in the sense of [4] or [6].

Remark 4. Observe that when $\varepsilon = 0$ the eigenvalues for equation (1) turn out to be explicit. As we mentioned in the introduction, this case was considered by D.L. Russell in [18]. But we can now observe, at least intuitively, that for small values of $\varepsilon > 0$ the behaviour of the spectrum proved in theorem 4.1 is just a perturbation of that limit and explicit case, $\varepsilon = 0$ (see figure 2, right figure, or [15] for details) .

Proof. (i) The proof of this part can be found in [15] and also in [14] with more details. The fact that the number of eigenvalues is infinite will be seen in part (iii) as a consequence of Picard's theorem.

(ii) In the new variable $z = \sqrt{1+\lambda\alpha}$, the characteristic equation can be expressed as

$$A_1(z)\,B_1(z) = A_2(z)\,B_2(z)$$

where

$$A_1(z) = \left(\frac{z^2 - 1}{\alpha} + \varepsilon z + \varepsilon r\right) \quad , \quad B_1(z) = e^{\frac{z^2 - 1}{\alpha z}}$$

$$A_2(z) = \left(\frac{z^2 - 1}{\alpha} - \varepsilon z + \varepsilon r\right) \quad , \quad B_2(z) = e^{-\frac{z^2 - 1}{\alpha z}}$$

We consider three partitions of the complex plane:

a) $\{|B_1(z)| < 1\}$, $\{|B_1(z)| = 1\}$ and $\{|B_1(z)| > 1\}$

b) $\{|B_2(z)| < 1\}$, $\{|B_2(z)| = 1\}$ and $\{|B_2(z)| > 1\}$

c) $\{|A_1(z)|^2 < |A_2(z)|^2\}$, $\{|A_1(z)|^2 = |A_2(z)|^2\}$ and $\{|A_1(z)|^2 > |A_2(z)|^2\}$

It is easy to see that these partitions do not depend on α, ε nor r except for the last one in the case $\varepsilon r \alpha < 1$ or $\varepsilon r \alpha \geq 1$. One can also see that the three partitions are compatible among them and with the relation

$$|A_1(z)|\,|B_1(z)| = |A_2(z)|\,|B_2(z)|$$

only if $z \in \{\mathrm{Re}\,(z) = 0\} \cup \{|z| < 1\}$ when $\varepsilon r \alpha \geq 1$, or only if $z \in \{\mathrm{Re}\,(z) = 0\} \cup \{\sqrt{1 - \varepsilon r \alpha} < |z| < 1\}$ when $\varepsilon r \alpha < 1$. Returning then to the variable $\lambda = \frac{z^2 - 1}{\alpha}$ we obtain our claim.

(iii) This part follows from the application of Picard's theorem to the equation

$$f(z) := e^{2\left(\frac{z^2 - 1}{\alpha z}\right)} \left(\frac{z^2 - 1 + \varepsilon \alpha z + \varepsilon \alpha r}{z^2 - 1 - \varepsilon \alpha z + \varepsilon \alpha r}\right) = 1$$

that is obtained from the characteristic equation (7) under the change of variable $z = \sqrt{1 + \lambda \alpha}$. We choose n_0 such that $f(z)$ does not have any root or singularity in D_{1/n_0} (the open disc centered on 0 and of radius $1/n_0$) apart from the essential singularity $z = 0$. We construct then the sequence of punctured discs:

$$E_k = D_{\frac{1}{n_0 + k}} \setminus \{0\} , \quad k \geq 0.$$

By definition, we have $E_{k+1} \subset E_k \subset \ldots \subset E_0$ for all $k > 0$ and $f(z)$ with no zeros and singularities in any of these E_k. By Picard's Great Theorem, there exist $z_{n_k} \in E_k$ with $f(z_{n_k}) = 1$ and, by construction of these discs, $\lim_{n_k \to \infty} z_{n_k} = 0$. So, the corresponding λ_{n_k} is a sequence of eigenvalues accumulating to $-1/\alpha$.

In the same way we can prove that there exists a sequence of eigenvalues accumulating to $-\infty$.

$\square$

Now we concentrate on the case of small and positive α. This case will be treated as a singular perturbation of the case $\alpha = 0$. The main tool to compare the spectra will be the concept of generalized convergence of operators of T. Kato in [8].

Lemma 4.2. $\|A_\alpha^{-1} - A_0^{-1}\|_{\mathcal{L}(\mathcal{H}, \mathcal{H})} \longrightarrow 0$ as $\alpha \longrightarrow 0$. Therefore, A_α converges to A_0 in the generalized sense as $\alpha \to 0$.

Proof. Writing A_α^{-1} and A_0^{-1} one can convince oneself that $A_\alpha^{-1}, A_0^{-1} \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ and

$$(A_\alpha^{\,-1} - A_0^{\,-1}) \begin{pmatrix} (u, u(1)) \\ (v, \beta) \end{pmatrix} = \begin{pmatrix} (-\alpha\, u, -\alpha\, u(1)) \\ (0, 0) \end{pmatrix}$$

if $\begin{pmatrix} (u, u(1)) \\ (v, \beta) \end{pmatrix} \in \mathcal{H}$. Then,

$$\left\| (A_\alpha^{\,-1} - A_0^{\,-1}) \begin{pmatrix} (w, w(1)) \\ (z, \beta) \end{pmatrix} \right\|^2 = \alpha^2 \int_0^1 |w_x|^2 \leq \alpha^2 \left\| \begin{pmatrix} (w, w(1)) \\ (z, \beta) \end{pmatrix} \right\|^2.$$

$\square$

Proof of theorem 1.2. By lemma 4.2 we know that A_α converges to A_0 when $\alpha \to 0$ in the generalized sense for closed operators. As it is seen in chapter IV.3.5 of T. Kato [8], this implies that bounded sets of the spectra depend continuously on α in a certain sense.

Theorems 4.1 and 2.1 allow us to say that all points of $\sigma(A_\alpha)$, $\alpha > 0$, are isolated and inside a sector that we call S_α, except for a unique point $\{-1/\alpha\}$. These results together with the generalized convergence theory imply that, if $\alpha > 0$ is small enough, we will certainly have a finite number of eigenvalues of A_α with real part greater than the rest of the spectrum. This happens because $-1/\alpha$ will be large (if α is small) and negative, and the eigenvalues of A_α will tend to those of A_0 that accumulate to the imaginary axis. So, for a fixed positive small α we have a finite maximal set of dominant eigenvalues.

Roughly speaking, we are going to see that the eigenvalues of A_α depend continuously on α and each one approaches a single one of A_0 as $\alpha \to 0$. Since the eigenvalues of A_0 approach the imaginary axis, the maximal set of dominant eigenvalues of A_α will necessarily change abruptly infinitely many times as α approaches 0.

This phenomenon can be seen in the figure 3. This figure represents the eigenvalues in the complex plane. The eigenvalues for $\alpha = 0$ are plotted with a star symbol, and those of $\alpha \neq 0$ with a circle symbol. Figure 3 left corresponds to $\alpha = 0.01$ and we see that the maximal dominant eigenvalues are $\lambda_2(\alpha)$ and $\overline{\lambda_2(\alpha)}$, while in figure 3 right, with $\alpha = 0.001$, they have been substituted by $\lambda_3(\alpha)$ and $\overline{\lambda_3(\alpha)}$. One can also expect that for some intermediate value of α the set of maximal dominant eigenvalues will have (at least) four elements.

To prove this, let us see first some properties of $\sigma(A_\alpha)$. Looking at the characteristic equation (7), we can check that the eigenvalues are the roots of an holomorphic function that depends continuously on α. So, each eigenvalue can be expressed as a continuous function $\lambda(\alpha)$, perhaps depending on an arbitrary choice at a non-simple root. If for each $\lambda(\alpha)$ there exists $M > 0$ such that

$$|\lambda(\alpha)| \leq M \text{ when } \alpha \to 0,$$

for a sequence $\alpha_n \to 0$ one will have that

$$\lambda(\alpha_n) \to \lambda_0, \text{ when } n \to \infty, \text{ and } \lambda_0 \in \sigma(A_0).$$

By using then the notion of generalized convergence, we will finally have that $\lambda(\alpha) \to \lambda_0$ as $\alpha \to 0$.

By contradiction, suppose now $|\lambda_k(\alpha)|$ is unbounded as $\alpha \to 0$. Suppose also that $\lambda_k(\alpha)$ belongs to the set of maximal dominant eigenvalues when $0 < \alpha \leq \alpha_k$, for a certain $\alpha_k > 0$. We are going to prove that this is not possible. Let us fix
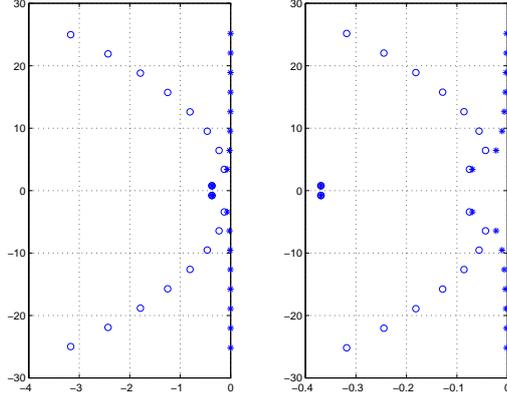
FIGURE 3. Change of the elements of the maximal dominant ei-
genvalues set.

$\delta > 0$. Because of the generalized convergence of A_α to A_0, the fact that $\sigma(A_0)$ approaches the imaginary axis and that $\lambda_k(\alpha)$ belongs to the maximal dominant set if $0 < \alpha \leq \alpha_k$, it must be satisfied that:

$$\operatorname{Re} \lambda_k(\alpha) > -\delta \quad \text{for } \alpha \text{ sufficiently small.}$$

Since we also know by theorem 4.1 that

$$\sigma(A_\alpha) \subset \left(-\infty, \frac{-1}{\alpha}\right] \cup \left\{\frac{1}{\alpha} - \varepsilon r < \left|\lambda + \frac{1}{\alpha}\right| < \frac{1}{\alpha}\right\},$$

it is easy to deduce that:

$$|\lambda_k(\alpha)| \leq \frac{\sqrt{2\,\delta}}{\sqrt{\alpha}}$$

if α is small enough. In particular, we have:

$$|\lambda_k(\alpha)| \to \infty \quad \text{and} \quad \alpha|\lambda_k(\alpha)| \to 0 \quad \text{as} \quad \alpha \to 0. \tag{8}$$

Using these limits in the characteristic equation (7) we can say that for all $\rho > 0$ there exists α_ρ such that:

$$\left| e^{\frac{2\,\lambda_k(\alpha)}{\sqrt{1+\lambda_k(\alpha)\alpha}}} - 1 \right| < \rho \quad \text{for all } 0 < \alpha < \alpha_\rho$$

By the properties of the exponential function, there exists $r(\rho) > 0$ such that:

$$\left| \frac{2\,\lambda_k(\alpha)}{\sqrt{1+\lambda_k(\alpha)\alpha}} - 2\pi i k_n \right| < r(\rho) \,, \quad 0 < \alpha < \alpha_\rho$$

with $k_n \in \mathbb{Z}$. Let us take $\rho > 0$ such that $r(\rho) < \pi$. For different k_n, the previous balls are disjoint. So, as $\lambda_k(\alpha)$ is continuous in $(0, \alpha_\rho)$, k_n must be the same for all $\alpha \in (0, \alpha_\rho)$. That is:

$$\left| \frac{2\,\lambda_k(\alpha)}{\sqrt{1+\lambda_k(\alpha)\alpha}} - 2\pi i k_0 \right| < r(\rho) \,, \quad 0 < \alpha < \alpha_\rho$$

for a certain $k_0 \in \mathbb{Z}$. Taking limits as $\alpha \to 0$ and using the limits seen in (8), we get into contradiction and, consequently, $\lambda_k(\alpha)$ must be bounded. And, from the previous argument, $\lambda_k(\alpha)$ tends to $\lambda_k(0) \in \sigma(A_0)$ as $\alpha \to 0$.

We are now ready to prove that the maximal set of dominant eigenvalues changes as $\alpha \to 0$. Suppose this is not true, that is suppose we have $\{\lambda_1(\alpha), \ldots, \lambda_l(\alpha)\}$ as the maximal set of dominant eigenvalues of A_α for all α small enough. As we have just proved, it happens that:

$$\lambda_i(\alpha) \to \lambda_i(0) \in \sigma(A_0) \ , \ i = 1, \ldots, l.$$

Let us fix $\mu > 0$ sufficiently small (for instance, $0 < \mu < \min_{i \in \{1, \ldots, l\}} \dfrac{|\operatorname{Re} \lambda_i(0)|}{4}$). Using the results from the theory of generalized convergence of T. Kato in [8], we can assure that:

$$|\lambda_i(\alpha) - \lambda_i(0)| < \mu \quad \text{for all} \ \ 0 < \alpha < \alpha_i$$

for each $i = 1, \ldots, l$. So let us take

$$\alpha_{min} = \min\{\alpha_1, \ldots, \alpha_l\} \ \ \text{and} \ \ K = \max_{i \in \{1, \ldots, l\}} \operatorname{Re} \lambda_i(\alpha_{min}) + \mu$$

which is strictly negative. We obviously have:

$$\operatorname{Re} \lambda_i(\alpha) < K < 0 \quad \forall \alpha \in (0, \alpha_{min}) \ , \ i = 1, \ldots, l.$$

As $\sigma(A_0)$ approaches the imaginary axis, it exists $\lambda^*(0) \in \sigma(A_0)$ such that:

$$\frac{K}{2} < \operatorname{Re} \lambda^*(0) < 0.$$

By generalized convergence again, there exists $\lambda^*(\alpha) \in \sigma(A_\alpha)$ with

$$|\lambda^*(\alpha) - \lambda^*(0)| < \frac{K}{2} \ \ \text{if} \ \ 0 < \alpha < \alpha^*.$$

Choosing α_0 small enough we have that $\lambda^*(\alpha)$ does not belong to the maximal set but $\operatorname{Re} \lambda^*(\alpha) > K > \operatorname{Re} \lambda_i(\alpha)$, if $0 < \alpha < \alpha_0$. This contradicts $\{\lambda_1(\alpha), \ldots, \lambda_l(\alpha)\}$ to be always the maximal set of dominant eigenvalues if α is small enough.

With this proof we have seen that the elements of the maximal dominant set cannot be the same as $\alpha \to 0$. Roughly speaking, as $\alpha \to 0$ the ones that were behind for larger α must jump in front of the previous maximal ones as they have to reach the imaginary axis. So, the maximal dominant ones are substituted again and again as $\alpha \to 0$. This argument also allows us to say that the number of maximal dominants cannot be constant as $\alpha \to 0$ either: at least it increases abruptly at the moment of these substitutions. $\qquad\square$

5. **The overdamped case.** In this section we are going to see that for some values of α, ε and r (α and ε large and r small) overdamping occurs: as the internal viscosity increases, solutions tend to zero at a slower rate, contradicting intuition. This is due to the fact that inner friction is so high that it becomes harder for the mass to move. This overdamping phenomenon is well known for ordinary differential equations of type (2), when d is large enough. For wave equations with weak damping, the corresponding phenomenon has also been observed (see, for instance, [1]). Moreover, in our case we will also see that this convergence to zero is free from oscillations because all the eigenvalues are real.

There is a particular limit case ($\varepsilon = \infty$, $r = 0$) in which the spectrum can be computed explicitly and where we can observe this overdamping for α large enough. In this case, the eigenvalues are:

$$\lambda_n^\pm = \frac{-(2n+1)^2\,\pi^2\,\alpha \pm \sqrt{(2n+1)^4\,\pi^4\,\alpha^2 - 16\,(2n+1)^2\,\pi^2}}{8} \quad,\quad n = 0, 1, 2, \ldots$$

If $\alpha > 4/\pi$ then the eigenvalues set consists of two sequences of strictly negative real numbers,

$$-\infty < \lambda_n^- < \lambda_n^+ < \frac{-1}{\alpha}$$

such that $\lambda_k^- \to -\infty$ and $\lambda_k^+ \to -1/\alpha$. So we see in this case that there is not a finite subset of dominant eigenvalues, that all the eigenvalues are real and that increasing α decreases the decay rate of solutions.

This explicit calculation induced us to believe that there may be an ε sufficiently large and an r sufficiently small for which the same situation is true when α is large enough. Theorem 1.3 assures that this happens.

Proof of theorem 1.3. As it has been seen in theorem 4.1, under the change of variables

$$z = \sqrt{1 + \lambda\alpha}$$

the eigenvalues λ of A_α are the roots of the transformed characteristic equation:

$$e^{2\,\frac{z^2-1}{\alpha z}} - \left(\frac{z^2 - 1 - \varepsilon\alpha z + \varepsilon\alpha r}{z^2 - 1 + \varepsilon\alpha z + \varepsilon\alpha r}\right) = 0. \tag{9}$$

And when $\varepsilon r\alpha < 1$ we have also seen that they are localized in

$$\{\operatorname{Re} z = 0\} \cup \{\sqrt{1 - \varepsilon r\alpha} < |z| < 1\}.$$

Inspired in the asymptotic behaviour of equation (9) when $r = 0$ and ε and α tend to ∞, our claim is that for r small enough and ε, α sufficiently large, equation (9) can be approximated by

$$e^{2\,\frac{z^2-1}{\alpha z}} = -1, \tag{10}$$

at least in some regions of the complex plane. Actually, we define:

$$g(z) = e^{2\,\frac{z^2-1}{\alpha z}} - \left(\frac{z^2 - 1 - \varepsilon\alpha z + \varepsilon\alpha r}{z^2 - 1 + \varepsilon\alpha z + \varepsilon\alpha r}\right)$$

and

$$f(z) = e^{2\,\frac{z^2-1}{\alpha z}} + 1$$

and we will prove that:

$$|f(z) - g(z)| < |f(z)| \quad \forall z \in \{\sqrt{1 - \varepsilon r\alpha} \le |z| \le 1\} \tag{11}$$

if ε, α are large and r is small enough. So we will obtain that $|g(z)| > 0 \quad \forall z \in \{\sqrt{1 - \varepsilon r\alpha} \le |z| \le 1\}$. This implies that $\sigma(A_\alpha) \subset (-\infty, -1/\alpha]$ for these values of the parameters. As $-1/\alpha$ is not an eigenvalue (see theorem 4.1), we conclude that for such values of ε, α, r the eigenvalues of A_α are all real and with real part strictly less than $-1/\alpha$.

So it only remains to see the inequality (11) to finish this proof. It has to be said that we are not interested in finding optimal bounds for the parameters, but only to prove that (11) holds. In order to do that, let us begin with bounding from

above the left hand side of the inequality. This can easily be done by writing the expression of the two funcions and we have that:

$$|f(z) - g(z)| < \left| \frac{2z^2 - 2 + 2\varepsilon r\alpha}{z^2 - 1 + \varepsilon r\alpha + \varepsilon z\alpha} \right|$$

If $r < 3/(4\varepsilon\alpha)$ and $\varepsilon\alpha > 8$ (observe that we have then $\varepsilon r\alpha < 1$ and, therefore, $0 < \sqrt{1 - \varepsilon r\alpha} \leq |z| \leq 1$) we can check, using usual bound rules, that:

$$|f(z) - g(z)| < \frac{1}{\varepsilon\alpha} \tag{12}$$

We have now to bound from below the right side of (11). Here we use usual exponential bounds to obtain that:

$$\left| e^{2\frac{z^2-1}{\alpha z}} + 1 \right| \geq 2 - \left| e^{2\frac{z^2-1}{\alpha z}} - 1 \right| \geq 3 - e^{\frac{8}{\alpha}}. \tag{13}$$

To join (12) and (13) we impose $\alpha > 8$. So, we have:

$$|f(z) - g(z)| < \frac{1}{\varepsilon\alpha} < 3 - e^{\frac{8}{\alpha}} < |f(z)|$$

if α is large enough (take $\alpha > 8$ for instance), ε is sufficiently large (take $\varepsilon > 8/\alpha$) and r is small enough (take $r < 3/(4\varepsilon\alpha)$). Hence, our proof is complete. $\qquad\square$

Remark 5. For these values of the parameters overdamping occurs, in the sense that increasing the inner damping does not increase the rate at which solutions tend to zero. Moreover, as there are not non-real eigenvalues, the solutions tend to zero without oscillations.

Corollary 1. *For some values of ε and r, and for α large enough, there does not exist a limit ODE for equation (1) in the sense explained in section 1.*

Proof. Theorem 1.3 implies that for these values of the parameters there is not a finite subset of dominant eigenvalues, because

$$-\infty < \lambda_n^+ < -\frac{1}{\alpha}$$

and $\lambda_n^+ \to -1/\alpha$, which is not an eigenvalue of A_α (see theorem 4.1). $\qquad\square$

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