

Lattices for 3-Dimensional Fuzzy Data generated by Fuzzy Galois Connections

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Abstract: Vagueness and high dimensional space data are usual features of current data. The paper is an approach to identify conceptual structures among fuzzy three dimensional data sets in order to get conceptual hierarchy. We propose a fuzzy extension of the Galois connections that allows to demonstrate an isomorphism theorem between fuzzy sets closures which is the basis for generating lattices ordered-sets.

Key-Words: Fuzzy Ternary Relations, Fuzzy Galois Connections, Fuzzy Closure Operators, Isomorphism Theorem, Lattice

1 Introduction

An important technique in Data Mining is Formal Concept Analysis [7, 18] which provides means to process tables which elements are 0 or 1 depending on an object verifies or not an attribute. When vagueness is introduced in the data we must replace classical logic by fuzzy logic so, the truth degrees of the data belong to $L \subset [0, 1]$ instead of $L = \{0, 1\}$. Several techniques deal with this kind of data composing the Fuzzy Formal Concept Analysis. Another reason of increment of complexity is when the dimensional data is greater than 2. In our approach we have replaced the usual 2-dimensional data table by a 3-dimensional data table.

The paper is organized as follows: in Section 2 we review the basic concepts of Formal Concept Analysis for objects and features, fuzzy ternary relations, projections and cylindrical extensions, in Section 3 we introduce the fuzzy Galois connections based on the α -cuts of the fuzzy ternary relation that defines the flood of vague data, in Section 4 we define fuzzy set closures and we demonstrate an isomorphism theorem, in Section 5 we construct the lattice ordered-set structure, and, finally, an example is shown in Section 6.

2 Preliminaries

Fuzzy Formal Concept Analysis provides a framework for designing hierarchies from relational information systems which data are represented by a table describing a fuzzy relation between a set of objects

and a set of attributes. Generalizing this framework to entries described by 3-dimensional tables means to suppose that the three sets involved are of different nature.

2.1 A binary Fuzzy Formal Concept Analysis approach

Given a set X we design by 2^X the set of all the subsets of X (it is also usual to design this set by $\wp(X)$). Let X be a set of objects, Y a set of attributes and R a binary relation between X and Y so $R \subset X \times Y$, then the induced operators are mappings

$$\uparrow : 2^X \longrightarrow 2^Y \text{ and } \downarrow : 2^Y \longrightarrow 2^X \text{ such that}$$

$$A^\uparrow = \{y : y \in Y \wedge \forall x \in A : (x, y) \in R\}$$

$$B^\downarrow = \{x : x \in X \wedge \forall y \in B : (x, y) \in R\}$$

A formal concept in (A, B, R) is a pair (A, B) such that $A^\uparrow = B$ and $B^\downarrow = A$. Therefore, a formal concept is a pair (A, B) such that B is the set of all attributes shared by all objects from A , and A is the set of all objects sharing all the attributes from B .

For a formal concept (A, B) , A is called an extent (set of objects covered by (A, B)) and B is called an intent (set of attributes covered by (A, B)).

We denote by $B(X, Y, R)$ the set of all formal concepts in (X, Y, R) , i.e,

$$B(X, Y, R) = \{(A, B) : A^\uparrow = B \wedge B^\downarrow = A\} \quad (1)$$

The aim is to provide $B(X, Y, R)$ by an order relation \leq such that $(B(X, Y, R), \leq)$ be a lattice called con-

cept lattice. The main definitions and theorems can be consulted in [7] and [18].

The data information table can be understood as a binary relation between objects and attributes, Wille's ideas and definitions have been extended to fuzzy environment simply considering the data belonging to the interval $[0, 1]$ and considering the data matrix as a fuzzy relation between objects and attributes [8, 9, 11, 24]. There are many approaches related to this work. Burusco and Fuentes-González were the first authors to generalize FCA to fuzzy formal contexts [4], Belohlavek and Pollandt employed the concept of resituated lattice [1, 2, 3, 16], Krajci and Yahia proposed the called "one-sided fuzzy approach" [12, 20] and, finally, Duráková et al proposed a procedure based on the a -cuts of the fuzzy relation [5].

Due to the fuzzy nature of the data we replace R by a fuzzy relation designed by the same symbol i.e. each pair of elements (x, y) has associated a membership value $\mu_R(x, y)$ that represents the degree to which an object x has an attribute y . In order to simplify the notation we substitute the expression $\mu_R(x, y)$ by $R(x, y)$. In these conditions the formal context is represented by a table which elements belong to $L \subset I = [0, 1]$ instead to be binary. Columns and rows define fuzzy subsets of objects and attributes. R can be understood as a fuzzy subset of $X \times Y$. We have already commented that many procedures are possible, we will follow our approach by means the a -cuts of R defined by

$${}^aR = \{(x, y) : (x, y) \in X \times Y \wedge R(x, y) \geq a\} \quad (2)$$

The information contained in these sets is equivalent to the information contained in R because

$$R(x, y) = \bigvee \{a : (x, y) \in {}^aR\} \quad (3)$$

We define the extent and intent sets associated to the a -cuts as follows

$$A^{\uparrow a} = \{y : y \in Y \wedge \forall x \in A : (x, y) \in {}^aR\} \quad (4)$$

$$B^{\downarrow a} = \{x : x \in X \wedge \forall y \in B : (x, y) \in {}^aR\} \quad (5)$$

Notice that this procedure is equivalent to define a boolean matrix $B_a = (b_{ij})$ where $b_{ij} = 1$ if $R(i, j) \geq a$ and $b_{ij} = 0$ otherwise. The following properties hold:

1. $\forall a, b \in I \ a \leq b$ then $A^{\uparrow a} \supset A^{\uparrow b}$ and $B^{\downarrow a} \supset B^{\downarrow b}$
2. $\forall A \in 2^X$ then $A \subset A^{\uparrow a \downarrow a}$
3. $\forall B \in 2^Y$ then $B \subset B^{\downarrow a \uparrow a}$

Moreover, Formal Concept Analysis and rough set theory are two related tools for analyzing information tables. Combining their strategies and results we achieve a more deepened way for dealing with this kind of information [10, 19, 21]. Recently some works have extended these new ideas to fuzzy data [17]. This methodology have been applied to many subjects as clustering [23, 6], information retrieval [13] or applied sciences [14, 15, 22].

2.2 Fuzzy Ternary Relations

Let $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_m\}$ and $Z = \{z_1, \dots, z_l\}$ be three finite sets. Let R be a fuzzy ternary relation between them with $R(x_i, y_j, z_k) = \alpha_{ijk}$. The most useful representation of this kind of relation consists in a finite set of 2-dimensional matrices, one for each element of a prefixed set [11], for instance if we select the set X we obtain

$$\begin{array}{c} \begin{array}{cccc} & z_1 & z_2 & \dots & z_l \\ y_1 & \left(\begin{array}{cccc} \alpha_{111} & \alpha_{112} & \dots & \alpha_{11l} \\ \alpha_{121} & \alpha_{122} & \dots & \alpha_{12l} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1m1} & \alpha_{1m2} & \dots & \alpha_{1ml} \end{array} \right) \\ y_2 & & & & \\ \vdots & & & & \\ y_m & & & & \end{array} \\ & x_1 \\ & \vdots \\ & z_1 & z_2 & \dots & z_l \\ y_1 & \left(\begin{array}{cccc} \alpha_{n11} & \alpha_{n12} & \dots & \alpha_{n1l} \\ \alpha_{n21} & \alpha_{n22} & \dots & \alpha_{n2l} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{nm1} & \alpha_{nm2} & \dots & \alpha_{nml} \end{array} \right) \\ y_2 & & & & \\ \vdots & & & & \\ y_m & & & & \end{array} \\ & x_n \end{array}$$

Similarly to Equation 6 the a -cuts of R are defined by

$${}^aR = \{(x, y, z) : R(x, y, z) \geq a\} \quad (6)$$

A practical example could be two sets of objects and one set of attributes where an entry would represent the opinion given by one element of Z about the truth degree in which an element of X verifies an attribute of Y . Obviously, many other cases are possible.

Usually a scientific research problem depends on several variables which induce us to ask about what would happen if the number of variables was smaller or greater. In our case we want to determine in which way is modified a 3-dimensional matrix with the exit of one variable and in which way is constructed a 3-dimensional matrix with the entrance of a new variable for a binary fuzzy relation. The mathematical

objects that help to study these kind of problems are projections, to decrease the number of variables; and cylindrical extensions, to increase the number of variables [11]. These concepts are applicable to any n -dimensional fuzzy relation but, obviously, we will present them for 3-dimensional fuzzy relations.

Definition 1 Let X, Y, Z be three finite sets and R a fuzzy relation defined in $X \times Y \times Z$. The projection of R in $X \times Y \times Z$ upon $X \times Y$ is a fuzzy relation in $X \times Y$ designed by $R \downarrow X \times Y$ such that

$$R \downarrow X \times Y(x, y) = \max_{u > (x, y)} R(u) \quad (7)$$

where $u = (x', y', z')$ and

$$(x', y', z') > (x, y) \quad \text{iff} \quad x' = x \text{ and } y' = y$$

Clearly, Equation (7) is equivalent to

$$R \downarrow X \times Y(x, y) = \max_{z \in Z} R(x, y, z)$$

Definition 2 Let X, Y, Z be three finite sets and R a fuzzy relation in $X \times Y$. The cylindrical extension $R \uparrow X \times Y \times Z - X \times Y$ of R is a fuzzy relation in $X \times Y \times Z$ such that

$$R \uparrow X \times Y \times Z - X \times Y(x, y, z) = R(x', y') \quad (8)$$

where (x', y') holds $(x, y, z) > (x', y')$

To relate projections and cylindric extensions by means of their compositions can help in understanding their meaning. Once again we simplify the notation defining

$$R \downarrow \uparrow = (R \downarrow X \times Y) \uparrow (X \times Y \times Z - X \times Y)$$

Where R is a fuzzy ternary relation defined in the cartesian product of X, Y and Z , and

$$R \uparrow \downarrow = (R \uparrow X \times Y \times Z - X \times Y) \downarrow X \times Y$$

Where R is a fuzzy binary relation defined in the cartesian product of X and Y .

Theorem 3 Let X, Y, Z be three finite sets. The following properties hold

(i) Let R a fuzzy relation in $X \times Y \times Z$ then

$$R \downarrow \uparrow (x, y, z) \geq R(x, y, z) \quad (9)$$

(ii) Let R be a fuzzy relation defined in $X \times Y$ then

$$R \uparrow \downarrow (x, y) = R(x, y) \quad (10)$$

Proof:

(i) $R \downarrow \uparrow (x, y, z)$ is a fuzzy ternary relation. For any x, y, z and applying Definition 1 and Definition 2 we have

$$\begin{aligned} R \downarrow \uparrow (x, y, z) &= R \downarrow (x, y) = \\ &= \max_{z' \in Z} R(x, y, z') \geq R(x, y, z) \end{aligned}$$

(ii) It is trivial from Definition 1 and Definition 2.

□

The second property is less relevant because the value of the membership function does not vary.

3 Fuzzy Galois Connections

Our aim is to generalize the concept of fuzzy Galois connection for binary fuzzy relations to fuzzy ternary relations. Our development is based on the a -cuts of R .

Definition 4 Let A, B, C be subsets of X, Y, Z and Δ, Γ, Θ subsets of $X \times Y, X \times Z, Y \times Z$ respectively. The fuzzy Galois connections are mappings defined by

$$A \uparrow_a^X = \{(y, z) \mid \forall x \in A \quad (x, y, z) \in {}^a R\} \quad (11)$$

$$B \uparrow_a^Y = \{(x, z) \mid \forall y \in B \quad (x, y, z) \in {}^a R\} \quad (12)$$

$$C \uparrow_a^Z = \{(x, y) \mid \forall z \in C \quad (x, y, z) \in {}^a R\} \quad (13)$$

$$\Delta \downarrow_a^{X \times Y} = \{z \mid \forall (x, y) \in \Delta \quad (x, y, z) \in {}^a R\} \quad (14)$$

$$\Gamma \downarrow_a^{X \times Z} = \{y \mid \forall (x, z) \in \Gamma \quad (x, y, z) \in {}^a R\} \quad (15)$$

$$\Theta \downarrow_a^{Y \times Z} = \{x \mid \forall (y, z) \in \Theta \quad (x, y, z) \in {}^a R\} \quad (16)$$

Remark that the symbology of the Galois connections, projection and cylindrical extension are similar but there is not confusion because the Galois connections are mappings between different sets and projections and cylindrical extensions are fuzzy relations. Notice that

$$A \uparrow_a^X \in 2^{Y \times Z} \text{ and } \Theta \downarrow_a^{Y \times Z} \in 2^X$$

$$B \uparrow_a^Y \in 2^{X \times Z} \text{ and } \Gamma \downarrow_a^{X \times Z} \in 2^Y$$

$$C \uparrow_a^Z \in 2^{X \times Y} \text{ and } \Delta \downarrow_a^{X \times Y} \in 2^Z$$

Therefore only the compositions between (11) with (16), (12) with (15) and (13) with (14) are possible. From this point of view we obtain three pairs of systems of 2-Galois connections namely,

$$\{(\uparrow_a^X, \downarrow_a^{Y \times Z}) \mid a \in L\}$$

$$\{(\uparrow_a^Y, \downarrow_a^{X \times Z}) \mid a \in L\}$$

$$\{(\uparrow_a^Z, \downarrow_a^{X \times Y}) \mid a \in L\}$$

From now on and in order to simplify the notation we will state the main results only for the first one, being absolutely similar for the remaining pairs of 2-Galois connections.

Lemma 5 For any $a, b \in L$, $A, A_1, A_2 \in 2^X$ and $\Theta \in 2^{Y \times Z}$ then

- (i) If $a \leq b$ then $A \uparrow_a^X \supset A \uparrow_b^X$ and $\Theta \downarrow_a^{Y \times Z} \supset \Theta \downarrow_b^{Y \times Z}$
- (ii) The set $\{a \in L \mid (y, z) \in \{x\} \uparrow_a^X\}$ contains a greatest element.
- (iii) $(y, z) \in A \uparrow_a^X$ iff $(y, z) \in \bigcap_{x \in A} \{x\} \uparrow_a^X$
- (iv) $A \subset A \uparrow_a^X \downarrow_a^{Y \times Z}$
- (v) $A_1 \subset A_2$ then $A_2 \uparrow_a^X \subset A_1 \uparrow_a^X$

Proof:

From the fact that L is finite and from Equations (6), (11) and (16) we have (i) and (ii). These statements mean that $\{(\uparrow_a^X, \downarrow_a^{Y \times Z})\}$ is L-nested.

(iii) If $(y, z) \in A \uparrow_a^X$ then the corresponding relational value is greater than a for any element of A then $(y, z) \in \bigcap_{x \in A} \{x\} \uparrow_a^X$, and viceversa.

$$(iv) A \uparrow_a^X \downarrow_a^{Y \times Z} = \{x \mid \forall (y, z) \in A \uparrow_a^X \ (x, y, z) \in {}^a R\} \supset A$$

because of Definition 1.

$$(v) A_1 \subset A_2 \text{ then } \bigcap_{x \in A_2} \{x\} \uparrow_a^X \subset \bigcap_{x \in A_1} \{x\} \uparrow_a^X \text{ and}$$

from (iii) conclude $A_2 \uparrow_a^X \subset A_1 \uparrow_a^X$ \square

Notice that in the same way (i), (ii), (iii), (iv) and (v) are true replacing \uparrow_a^X by $\downarrow_a^{Y \times Z}$.

4 Fuzzy Closures

Lemma 6 For any $A \in 2^X$

$$A \uparrow_a^X \downarrow_a^{Y \times Z} \uparrow_a^X = A \uparrow_a^X$$

Proof: We will prove it by proving both inclusions

- (i) Combining (iv) and (v) of Lemma 5 we get $A \uparrow_a^X \downarrow_a^{Y \times Z} \uparrow_a^X \subset A \uparrow_a^X$.
- (ii) On the other hand, suppose that exists $(y^*, z^*) \in A \uparrow_a^X$ but $(y^*, z^*) \notin A \uparrow_a^X \downarrow_a^{Y \times Z} \uparrow_a^X$

then exists $x \in A \uparrow_a^X \downarrow_a^{Y \times Z}$ such that $R(x, y^*, z^*) < a$.

As $x \in A \uparrow_a^X \downarrow_a^{Y \times Z}$ then, for any $(y, z) \in A \uparrow_a^X$ it holds $R(x, y, z) \geq a$ In particular for $(y^*, z^*) \in A \uparrow_a^X$ then $R(x, y^*, z^*) \geq a$ what is contradictory. \square

Lemma 7 Let $A, B \in 2^X$ then

$$\text{If } A \subset B \text{ then } A \uparrow_a^X \downarrow_a^{Y \times Z} \subset B \uparrow_a^X \downarrow_a^{Y \times Z}$$

Proof: It is a consequence of applying two times statement (v) of Lemma 5. \square

Lemma 8 The composition mapping

$$\uparrow_a^X \downarrow_a^{Y \times Z} : 2^X \rightarrow 2^X$$

is called a fuzzy closure operator in the sense that holds

- (i) $A \subset A \uparrow_a^X \downarrow_a^{Y \times Z}$
- (ii) $A \uparrow_a^X \downarrow_a^{Y \times Z} (\uparrow_a^X \downarrow_a^{Y \times Z}) = A \uparrow_a^X \downarrow_a^{Y \times Z}$

Proof:

(i) Proved in Lemma 5.

(ii) From Lemma 5 and Lemma 7 we get $A \uparrow_a^X \downarrow_a^{Y \times Z} \subset A \uparrow_a^X \downarrow_a^{Y \times Z} (\uparrow_a^X \downarrow_a^{Y \times Z})$

On the other hand, from Lemma 5 and Lemma 6 we get

$$A \uparrow_a^X \downarrow_a^{Y \times Z} (\uparrow_a^X \downarrow_a^{Y \times Z}) \subset A \uparrow_a^X \downarrow_a^{Y \times Z} \quad \square$$

Definition 9

$$\Omega(X, \uparrow_a^X \downarrow_a^{Y \times Z}) = \{A \mid A = A \uparrow_a^X \downarrow_a^{Y \times Z}\} \quad (17)$$

namely, is the fuzzy set closure by $\uparrow_a^X \downarrow_a^{Y \times Z}$.

Theorem 10

$$\Omega(X, \uparrow_a^X \downarrow_a^{Y \times Z}) = \bigcup_{\Theta \in 2^{Y \times Z}} \Theta \downarrow_a^{Y \times Z} \quad (18)$$

and its dual formula

$$\Omega(Y \times Z, \downarrow_a^{Y \times Z} \uparrow_a^X) = \bigcup_{A \subset X} A \uparrow_a^X \quad (19)$$

Proof: We will prove both inclusions.

(i) If $A \in \bigcup_{\Theta \in 2^{Y \times Z}} \Theta \downarrow_a^{Y \times Z}$ then

Exists $\Theta \in 2^{Y \times Z}$ such that $A = \Theta \downarrow_a^{Y \times Z}$ and applying Lemma 6 we get

$$A \uparrow_a^X \downarrow_a^{Y \times Z} = A.$$

Therefore A is a closure by $\uparrow_a^X \downarrow_a^{Y \times Z}$.

(ii) On the other hand,

If $A \in \Omega(X, \uparrow_a^X \downarrow_a^{Y \times Z})$ then $A = A \uparrow_a^X \downarrow_a^{Y \times Z}$ and then $A = (A \uparrow_a^X) \downarrow_a^{Y \times Z}$. \square

This result allows us calculating the fuzzy set closure in a simple way.

Next result relates the fuzzy Galois connections of a ternary fuzzy relation R and the composition of projection and cylindrical extension $R \downarrow \uparrow$. In order to simplify the notation we design by $A^* \uparrow_a^X$ the fuzzy Galois connections with respect $R \downarrow \uparrow$.

Theorem 11 For any $A \in 2^X$

$$A \uparrow_a^X \subset A^* \uparrow_a^X \quad (20)$$

Proof: It is a consequence of applying Equations 6, 9 and 11. \square

Theorem 12

$$\Omega(Y \times Z, \downarrow_a^{Y \times Z} \uparrow_a^X) \simeq \Omega(X, \uparrow_a^X \downarrow_a^{Y \times Z}) \quad (21)$$

Where \simeq means a bijective map.

Proof: The bijective map is defined by

$$\begin{aligned} \Omega(Y \times Z, \downarrow_a^{Y \times Z} \uparrow_a^X) &\rightarrow \Omega(X, \uparrow_a^X \downarrow_a^{Y \times Z}) \\ A \uparrow_a^X &\rightarrow (A \uparrow_a^X) \downarrow_a^{Y \times Z} \end{aligned} \quad (22)$$

This map is well defined due to Theorem 10.

Injective

If $(A_1 \uparrow_a^X) \downarrow_a^{Y \times Z} = (A_2 \uparrow_a^X) \downarrow_a^{Y \times Z}$ then

$$(A_1 \uparrow_a^X) \downarrow_a^{Y \times Z} \uparrow_a^X = (A_2 \uparrow_a^X) \downarrow_a^{Y \times Z} \uparrow_a^X$$

and applying Lemma 6 we have

$$A_1 \uparrow_a^X = A_2 \uparrow_a^X.$$

Surjective

For any $A \in \Omega(X, \uparrow_a^X \downarrow_a^{Y \times Z})$ exists $\Theta \in 2^{Y \times Z}$ such that

$A = \Theta \downarrow_a^{Y \times Z}$, then, applying Lemma 6

$\Theta \downarrow_a^{Y \times Z} \uparrow_a^X \downarrow_a^{Y \times Z} = A$, and, finally

$$A = (\Theta \downarrow_a^{Y \times Z} \uparrow_a^X) \downarrow_a^{Y \times Z}$$

Where, obviously,

$$\Theta \downarrow_a^{Y \times Z} \uparrow_a^X \in \Omega(Y \times Z, \downarrow_a^{Y \times Z} \uparrow_a^X). \quad \square$$

This theorem is an important result because allow us to associate in an unique way elements of different closures.

5 Lattice ordered-set

A partial ordered set (or poset) is a set taken together with a partial order on it. Formally, a partial ordered set is defined as an ordered pair $P = (A, \leq)$, where A is called the ground set of P and \leq is the partial order of P .

The infimum is the greatest lower bound of a set S . The supremum is the least upper bound of a set S .

A Lattice ordered-set is a poset in which each two-element subset $\{a, b\}$ has an infimum, denoted $\inf\{a, b\}$, and a supremum, denoted $\sup\{a, b\}$.

A lattice (L, \wedge, \vee) can be obtained from a a lattice-ordered poset (L, \leq) by defining

$$a \wedge b = \inf\{a, b\} \quad \text{and} \quad a \vee b = \sup\{a, b\}$$

In our case and due to the results of Section 4 a structure of lattice can be given to the cartesian product of the fuzzy closures, namely

$$\Omega(Y \times Z, \downarrow_a^{Y \times Z} \uparrow_a^X) \times \Omega(X, \uparrow_a^X \downarrow_a^{Y \times Z})$$

as follows

$$\begin{aligned} (A_1, \Theta_1) \leq_1 (A_2, \Theta_2) &\text{ iff} \\ \text{iff } A_1 \subset A_2 \text{ and } \Theta_2 \subset \Theta_1 &\end{aligned} \quad (23)$$

or its dual

$$\begin{aligned} (A_1, \Theta_1) \leq_2 (A_2, \Theta_2) &\text{ iff} \\ \text{iff } A_2 \subset A_1 \text{ and } \Theta_1 \subset \Theta_2 &\end{aligned} \quad (24)$$

Notice that for any other system of 2-Galois connections we would obtain another lattice. Therefore we can get three systems of lattices depending on the a -cut of the fuzzy relation R .

6 Example

Let X, Y and Z be three finite sets with cardinals $|X| = 3, |Y| = 2$ and $|Z| = 3$. We represent their elements by $X = \{x_1, x_2, x_3\}, Y = \{y_1, y_2\}$ and $Z = \{z_1, z_2, z_3\}$. Let R be a ternary fuzzy relation represented by three 2-dimensional matrices everyone for each element of X as follows

$$\begin{array}{c} z_1 \quad z_2 \quad z_3 \\ y_1 \begin{pmatrix} 0.7 & 0.4 & 0.6 \\ 0.3 & 0.8 & 0.7 \end{pmatrix} \\ x_1 \end{array}$$

$$\begin{array}{c} z_1 \quad z_2 \quad z_3 \\ y_1 \begin{pmatrix} 0.5 & 0.9 & 0.8 \\ 0.7 & 0.6 & 0.4 \end{pmatrix} \\ x_2 \end{array}$$

$$\begin{array}{c} z_1 \quad z_2 \quad z_3 \\ y_1 \begin{pmatrix} 0.7 & 0.3 & 0.9 \\ 1 & 0.6 & 0.2 \end{pmatrix} \\ x_3 \end{array}$$

In order to obtain the closure we need to calculate $A \uparrow_a^X \downarrow_a^{Y \times Z}$ and $\Theta \downarrow_a^{Y \times Z} \uparrow_a^X$ for each element of X and $Y \times Z$.

For instance, if $0.6 < a \leq 0.7$

$$\begin{aligned} \{x_1, x_2\} \uparrow_a^X \downarrow_a^{Y \times Z} &= \{x_1, x_2, x_3\} \text{ and} \\ \{(y_1, z_1), (y_2, z_1)\} \downarrow_a^{Y \times Z} \uparrow_a^X &= \\ &= \{(y_1, z_1), (y_1, z_3), (y_2, z_1), (y_2, z_2)\} \end{aligned}$$

and so on.

From Equations (18) and (19) we obtain

$$\begin{aligned} \Omega(X, \uparrow_a^X \downarrow_a^{Y \times Z}) &= \\ &= \{\emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_3\}, \{x_2, x_3\}, \\ &\quad \{x_1, x_2, x_3\}\} \end{aligned}$$

$$\begin{aligned} \Omega(Y \times Z, \downarrow_a^{Y \times Z} \uparrow_a^X) &= \\ &= \{\emptyset, \{(y_2, z_2), (y_1, z_3)\}, \\ &\quad \{(y_1, z_1), (y_1, z_3), (y_2, z_2)\}, \{(y_1, z_3), (y_2, z_1), (y_2, z_2)\}, \\ &\quad \{(y_1, z_1), (y_2, z_1), (y_2, z_2), (y_1, z_3)\}, \\ &\quad \{(y_1, z_1), (y_1, z_3), (y_2, z_2), (y_2, z_3)\}, \\ &\quad \{(y_1, z_2), (y_1, z_3), (y_2, z_1), (y_2, z_2)\}, F \times G\} \end{aligned}$$

From Equation (22) we obtain the following bijective application between the previous two sets of closures which will give us the lattice structure.

$$\begin{aligned} f_1 : \Omega(E, g_1) &\longrightarrow \Omega(F \times G, g_{23}) \\ \emptyset &\longrightarrow F \times G \\ \{x_1\} &\longrightarrow \{(y_1, z_1), (y_1, z_3), (y_2, z_2), (y_2, z_3)\} \\ \{x_2\} &\longrightarrow \{(y_1, z_2), (y_1, z_3), (y_2, z_1), (y_2, z_2)\} \\ \{x_3\} &\longrightarrow \{(y_1, z_1), (y_2, z_1), (y_2, z_2), (y_1, z_3)\} \\ \{x_1, x_3\} &\longrightarrow \{(y_1, z_1), (y_1, z_3), (y_2, z_2)\} \\ \{x_2, x_3\} &\longrightarrow \{(y_1, z_3), (y_2, z_1), (y_2, z_2)\} \\ \{x_1, x_2, x_3\} &\longrightarrow \{(y_2, z_2), (y_1, z_3)\} \end{aligned}$$

Pairs consisting of one element and its image will be the nodes of the lattice defined by orderings of Equations (23) and (24).

6.1 Set of lattices for R depending on the a -cuts

The lattice for the different values of the a -cuts are represented in Fig. 1, . . . ,9. Notice that in any node of the net there is a pair of two sets. The first one is a subset of X and the second one a subset of $Y \times Z$ which are image and anti-image for the isomorphism between the two closures. In order to simplify the notation we introduce the symbol $F_i = (A_i, \Theta_i)$, being A_i the image of Θ_i by the isomorphism (22). As we have already discussed we could obtain two more lattices for the other two isomorphisms.

$$\begin{aligned} \text{For instance, for } 0.6 < a \leq 0.7, \\ A_1 &= \emptyset, A_2 = \{x_1\}, A_3 = \{x_2\}, \\ A_4 &= \{x_3\}, A_5 = \{x_1, x_3\}, A_6 = \{x_2, x_3\}, \\ A_7 &= \{x_1, x_2, x_3\}, \Theta_1 = Y \times Z, \\ \Theta_2 &= \{(y_1, z_1), (y_1, z_3), (y_2, z_2), (y_2, z_3)\}, \\ \Theta_3 &= \{(y_1, z_2), (y_1, z_3), (y_2, z_1), (y_2, z_2)\}, \\ \Theta_4 &= \{(y_1, z_1), (y_2, z_1), (y_2, z_2), (y_1, z_3)\}, \\ \Theta_5 &= \{(y_1, z_1), (y_1, z_3), (y_2, z_2)\}, \\ \Theta_6 &= \{(y_1, z_3), (y_2, z_1), (y_2, z_2)\}, \\ \Theta_7 &= \{(y_2, z_2), (y_1, z_3)\}, \end{aligned}$$

Moreover, we introduce the notation $\mathcal{L}_{a_1}^{a_2}$ for the lattice with the a -cut verifying $a_1 < a \leq a_2$. Now we present the lattice structure in function on the value of the a -cut.

Lattice $\mathcal{L}_0^{0.2}$

$$A_1 = X, \Theta_1 = Y \times Z$$

The lattice reduces to a unique element F_1 and is represented in Figure 1.



Figure 1: Lattice $\mathcal{L}_0^{0.2}$

Lattice $\mathcal{L}_{0.2}^{0.3}$

$$\begin{aligned} A_1 &= \{x_1, x_2\}, \Theta_1 = Y \times Z, \\ A_2 &= \{x_1, x_2, x_3\}, \\ \Theta_2 &= \{(y_1, z_1), (y_1, z_2), (y_2, z_1), (y_2, z_2), (y_1, z_3)\} \end{aligned}$$

The lattice is represented in Figure 2.



Figure 2: Lattice $\mathcal{L}_{0.2}^{0.3}$

Lattice $\mathcal{L}_{0.3}^{0.4}$

- $A_1 = \{x_2\},$
- $\Theta_1 = Y \times Z$
- $A_2 = \{x_1, x_2\},$
- $\Theta_2 = \{(y_1, z_1), (y_1, z_2), (y_1, z_3), (y_2, z_2), (y_2, z_3)\}$
- $A_3 = \{x_2, x_3\},$
- $\Theta_3 = \{(y_1, z_1), (y_1, z_3), (y_2, z_1), (y_2, z_2)\}$
- $A_4 = \{x_1, x_2, x_3\},$
- $\Theta_4 = \{(y_1, z_1), (y_2, z_2), (y_1, z_3)\}$

The lattice is represented in Figure 3. Notice that at this level we loose the property of total order.

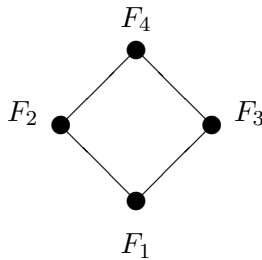


Figure 3: Lattice $\mathcal{L}_{0.3}^{0.4}$

Lattice $\mathcal{L}_{0.4}^{0.5}$

- $A_1 = \emptyset,$
- $\Theta_1 = Y \times Z$
- $A_2 = \{x_1\}$
- $\Theta_2 = \{(y_1, z_1), (y_1, z_3), (y_2, z_2), (y_2, z_3)\}$
- $A_3 = \{x_2\}$
- $\Theta_3 = \{(y_1, z_1), (y_1, z_2), (y_1, z_3), (y_2, z_1), (y_2, z_2)\}$
- $A_4 = \{x_2, x_3\}$
- $\Theta_4 = \{(y_1, z_1), (y_1, z_3), (y_2, z_1), (y_2, z_2)\}$
- $A_5 = \{x_1, x_2, x_3\}$
- $\Theta_5 = \{(y_1, z_1), (y_1, z_3), (y_2, z_2)\}$

The lattice is represented in Figure 4.

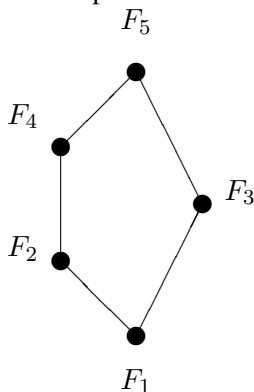


Figure 4: Lattice $\mathcal{L}_{0.4}^{0.5}$

Lattice $\mathcal{L}_{0.5}^{0.6}$

This case has been described at the beginning of the section. The lattice is represented in Figure 5.

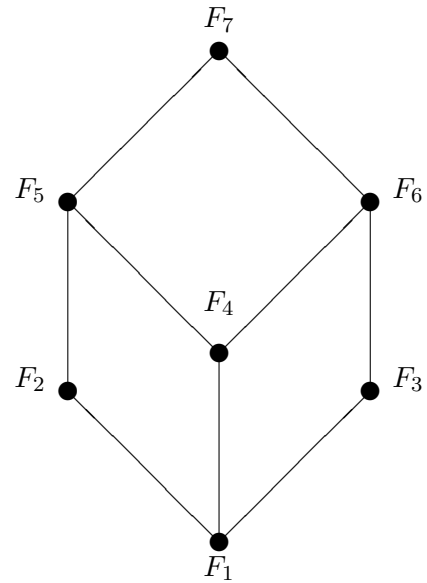


Figure 5: Lattice $\mathcal{L}_{0.5}^{0.6}$

Lattice $\mathcal{L}_{0.6}^{0.7}$

- $A_1 = \emptyset,$
- $\Theta_1 = \Theta_1 = Y \times Z$
- $A_2 = \{x_1\}$
- $\Theta_2 = \{(y_1, z_1), (y_2, z_2), (y_2, z_3)\}$
- $A_3 = \{x_2\}$
- $\Theta_3 = \{(y_1, z_2), (y_1, z_3), (y_2, z_1)\}$
- $A_4 = \{x_3\}$
- $\Theta_4 = \{(y_1, z_1), (y_1, z_3), (y_2, z_1)\}$
- $A_5 = \{x_1, x_3\},$
- $\Theta_5 = \{(y_1, z_1)\}$
- $A_6 = \{x_2, x_3\},$
- $\Theta_6 = \{(y_1, z_3), (y_2, z_1)\}$
- $A_7 = \{x_1, x_2, x_3\},$
- $\Theta_7 = \emptyset$

The lattice is represented in Figure 6.

Lattice $\mathcal{L}_{0.7}^{0.8}$

- $A_1 = \emptyset,$
- $\Theta_1 = Y \times Z$
- $A_2 = \{x_1\},$
- $\Theta_2 = \{(y_2, z_2)\}$
- $A_3 = \{x_2\},$
- $\Theta_3 = \{(y_1, z_2), (y_1, z_3)\}$
- $A_4 = \{x_3\}$
- $\Theta_4 = \{(y_1, z_3), (y_2, z_1)\}$
- $A_5 = \{x_2, x_3\},$
- $\Theta_5 = \{(y_1, z_3)\}$
- $A_6 = \{x_1, x_2, x_3\},$
- $\Theta_6 = \emptyset$

The lattice is represented in Figure 7.

Lattice $\mathcal{L}_{0.8}^{0.9}$

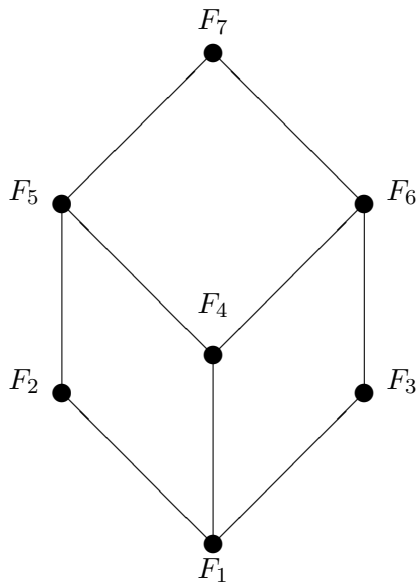


Figure 6: Lattice $\mathcal{L}_{0.6}^{0.7}$

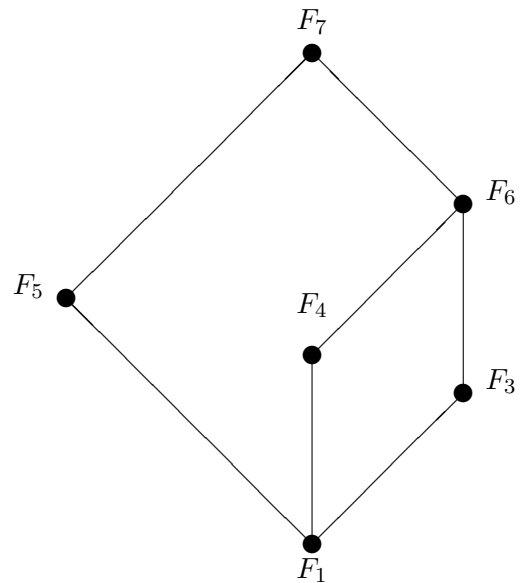


Figure 7: Lattice $\mathcal{L}_{0.7}^{0.8}$

- $A_1 = \emptyset,$
- $\Theta_1 = Y \times Z$
- $A_2 = \{x_2\},$
- $\Theta_2 = \{(y_1, z_2)\}$
- $A_3 = \{x_3\},$
- $\Theta_3 = \{(y_1, z_3), (y_2, z_1)\}$
- $A_4 = \{x_1, x_2, x_3\},$
- $\Theta_4 = \emptyset$

The lattice is represented in Figure 8.

Lattice $\mathcal{L}_{0.9}^1$

- $A_1 = \emptyset,$
- $\Theta_1 = Y \times Z$
- $A_2 = \{x_3\},$
- $\Theta_2 = \{(y_2, z_1)\}$
- $A_3 = \{x_1, x_2, x_3\},$
- $\Theta_3 = \emptyset$

The lattice is represented in Figure 9.

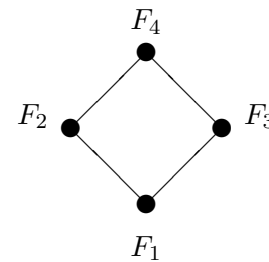


Figure 8: Lattice $\mathcal{L}_{0.8}^{0.9}$

we get the matrix

$$\begin{matrix} & y_1 & y_2 \\ x_1 & \begin{pmatrix} 0.7 & 0.8 \end{pmatrix} \\ x_2 & \begin{pmatrix} 0.9 & 0.7 \end{pmatrix} \\ x_3 & \begin{pmatrix} 0.9 & 1 \end{pmatrix} \end{matrix}$$

For this binary fuzzy relation we obtain

Lattice $\mathcal{L} \downarrow_0^{0.7}$

$$A_1 = X, \Theta_1 = Y$$

The lattice is represented in Figure 10.

Lattice $\mathcal{L} \downarrow_{0.7}^{0.8}$

- $A_1 = X,$
- $\Theta_1 = \emptyset$
- $A_2 = \{x_1, x_3\},$
- $\Theta_2 = \{y_2\}$
- $A_3 = \{x_2, x_3\},$
- $\Theta_3 = \{y_1\}$
- $A_4 = \{x_3\},$
- $\Theta_4 = \{y_1, y_2\}$

6.2 Set of lattices for $R \downarrow$ depending on the a -cuts

We determine $R \downarrow (X \times Y)$, projection of R above $X \times Y$ using Equation 7, for instance

$$\begin{aligned} R \downarrow (X \times Y)(x_1, y_1) &= \\ &= \max\{R(x_1, y_1, z_1), R(x_1, y_1, z_2), R(x_1, y_1, z_3)\} = \\ &= \max\{0.7, 0.4, 0.6\} = 0.7 \end{aligned}$$

In the same way we achieve all the other values of the membership function; at the end of the process

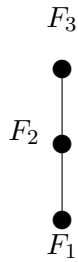


Figure 9: Lattice $\mathcal{L}_{0.9}^1$



Figure 10: Lattice $\mathcal{L}_{0.7}^1$

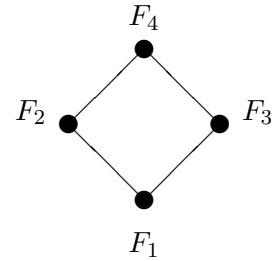


Figure 11: Lattice $\mathcal{L}_{0.7}^{0.8}$

The lattice is represented in Figure 11.

Lattice $\mathcal{L}_{0.8}^{0.9}$

$$A_1 = X,$$

$$\Theta_1 = \emptyset$$

$$A_2 = \{x_2, x_3\},$$

$$\Theta_2 = \{y_1\}$$

$$A_3 = \{x_3\},$$

$$\Theta_3 = \{y_1, y_2\}$$

The lattice is represented in Figure 12.

Lattice $\mathcal{L}_{0.9}^1$

$$A_1 = X,$$

$$\Theta_1 = \emptyset$$

$$A_2 = \{x_3\},$$

$$\Theta_2 = \{y_2\}$$

$$A_3 = \emptyset,$$

$$\Theta_3 = \{y_1, y_2\}$$

The lattice is represented in Figure 13.

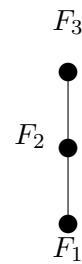


Figure 12: Lattice $\mathcal{L}_{0.8}^{0.9}$

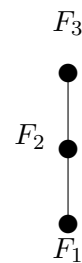


Figure 13: Lattice $\mathcal{L}_{0.9}^1$

6.3 Set of lattices for $R \downarrow \uparrow$ depending on the α -cuts

Calculating the cylindrical extension in $X \times Y \times Z$ of the previous fuzzy relation, we get

$$R \downarrow \uparrow (x_i, y_j, z_k) = R \downarrow (x_i, y_k)$$

The new tridimensional matrix is:

$$\begin{matrix}
 & z_1 & z_2 & z_3 \\
 y_1 & (0.7 & 0.7 & 0.7) \\
 y_2 & (0.8 & 0.8 & 0.8)
 \end{matrix}$$

$$\begin{matrix}
 & x_1 & & \\
 & z_1 & z_2 & z_3 \\
 y_1 & (0.9 & 0.9 & 0.9) \\
 y_2 & (0.7 & 0.7 & 0.7)
 \end{matrix}$$

$$\begin{matrix}
 & x_2 & & \\
 & z_1 & z_2 & z_3 \\
 y_1 & (0.9 & 0.9 & 0.9) \\
 y_2 & (1 & 1 & 1)
 \end{matrix}$$

As is a tridimensional matrix we will make the same study that for the original tridimensional relation.

Lattice $\mathcal{L} \downarrow_{0.7}^{\uparrow 0.7}$

$$A_1 = X, \Theta_1 = Y \times Z$$

The lattice is represented in Figure 14.

Lattice $\mathcal{L} \downarrow_{0.7}^{\uparrow 0.8}$

$$A_1 = \{x_3\},$$

$$\Theta_1 = Y \times Z$$

$$A_2 = \{x_1, x_3\}$$

$$\Theta_2 = \{(y_2, z_1), (y_2, z_2), (y_2, z_3)\}$$

$$A_3 = \{x_2, x_3\}$$

$$\Theta_3 = \{(y_1, z_1), (y_1, z_2), (y_1, z_3)\}$$

$$A_4 = X,$$

$$\Theta_4 = \emptyset$$

The lattice is represented in Figure 15.

Lattice $\mathcal{L} \downarrow_{0.8}^{\uparrow 0.9}$

$$A_1 = \{x_3\},$$

$$\Theta_1 = Y \times Z$$

$$A_2 = \{x_2, x_3\}$$

$$\Theta_2 = \{(y_1, z_1), (y_1, z_2), (y_1, z_3)\}$$

$$A_3 = X,$$

$$\Theta_3 = \emptyset$$

The lattice is represented in Figure 16.

Lattice $\mathcal{L} \downarrow_{0.9}^{\uparrow 1}$

$$A_1 = \emptyset,$$

$$\Theta_1 = Y \times Z$$

$$A_2 = \{x_3\}$$

$$\Theta_2 = \{(y_2, z_1), (y_2, z_2), (y_2, z_3)\}$$

$$A_3 = X,$$

$$\Theta_3 = \emptyset$$

The lattice is represented in Figure 17.

7 Conclusion

The results explained in the previous sections show that in the domain of Formal Concept Analysis the main definitions and results for binary 2-dimensional data can be translate to fuzzy 3-dimensional data in



Figure 14: Lattice $\mathcal{L} \downarrow_{0.7}^{\uparrow 0.7}$

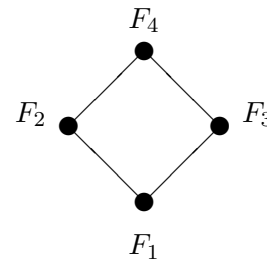


Figure 15: Lattice $\mathcal{L} \downarrow_{0.7}^{\uparrow 0.8}$

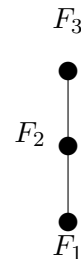


Figure 16: Lattice $\mathcal{L} \downarrow_{0.8}^{\uparrow 0.9}$

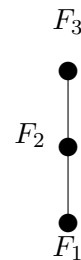


Figure 17: Lattice $\mathcal{L} \downarrow_{0.9}^{\uparrow 1}$

function of the a -cuts obtaining similar results. Moreover, it is possible to reduce the number of variables by means of projecting in a subspace and increase the number by means of the cylindrical extension. Combining these operations we loose information as expected.

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