Space Manifold dynamics

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The term ***Space Manifold Dynamics (SMD)*** has been proposed for encompassing the various applications of Dynamical Systems methods to spacecraft mission analysis and design, ranging from the exploitation of libration orbits around the collinear Lagrangian points to the design of optimal station-keeping and eclipse avoidance manoeuvres or the determination of low energy lunar and interplanetary transfers.
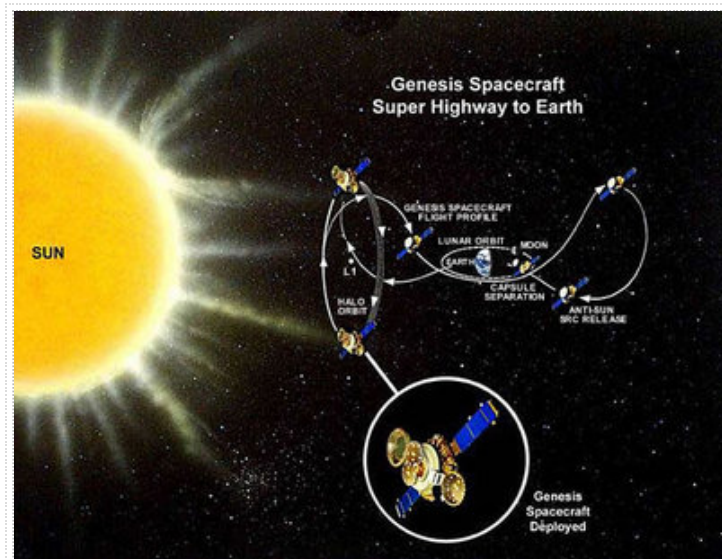

Figure 1: The complicated Genesis mission trajectory was the first one completely designed using Space Manifold Dynamics (Credit: NASA/JPL-Caltech.)

Introduction

The classical approach to interplanetary spacecraft trajectory design used, for instance, for the Apollo lunar missions, for the Voyagers grand tour, or for the New Horizons mission, has been based on patched-conics transfers and planetary swing-bys followed by numerical optimization procedures.

The convenience of the new approach using space manifold dynamics arises due to space missions that require new and unusual kinds of orbits to meet their goals, which cannot be found combining one or more Keplerian orbits, and also have low fuel requirements for the execution of transfer and station keeping manoeuvres. Dynamical Systems tools allow the analysis of the natural dynamics of the problem in a systematic way, and can be used, for instance, to design new "low-energy" orbits, cheap station keeping procedures and eclipse avoidance strategies, fitting the required mission constraints.

The starting point of this new approach was the SOHO (http://en.wikipedia.org/wiki/Solar_and_Heliospheric_Observatory) mission (Gómez et al., 2001). This was the second libration point mission and pursued studies of the Earth-Sun interactions, in a first step of what now is known as Space Weather. SOHO was the successor of the ground-breaking mission performed by the third International Sun-Earth Explorer spacecraft ISEE--3 (http://en.wikipedia.org/wiki/International_Cometary_Explorer) (Dunham and Farquhar, 2003).

After ISEE-3 and SOHO, interest in the scientific advantages of the Lagrange libration points for space missions has continued to increase and inspired even more challenging objectives that are reflected in MAP (http://en.wikipedia.org/wiki/Wilkinson_Microwave_Anisotropy_Probe) , Genesis (http://en.wikipedia.org/wiki/Genesis_(spacecraft)) , and Herschel--Plank (http://en.wikipedia.org/wiki/Planck_(spacecraft)) missions. Genesis was the first mission whose trajectory was completely designed using only Dynamical Systems tools (Howell et al., 2001) .

The simplest dynamical model used for this new approach is the Circular Restricted Three-Body Problem (CRTBP). Most of the current applications of Space Manifold Dynamics have been obtained thanks to the analysis of the dynamics in the vicinity of its equilibrium points, mainly the two collinear libration points L_1 and L_2 . In this article we will focus our attention in these two points. Although the qualitative dynamical behaviour of the L_3 collinear point is close to the one of L_1 and L_2 , this is not the case from a quantitative point of view; even some of the procedures that can be used for the analysis of the dynamics in a neighborhood of $L_{1,2}$ do not apply to L_3 .

Fundamentals

Hamiltonian systems provide the general setting for the analysis and the developing of methodologies for the Space Manifold Dynamics. For the design of space missions, for which the gravitational influence on the spacecraft of at least two main bodies must be taken into account, the CRTBP is the natural model to start with. Nevertheless, the same kind of dynamics and structures can be found around any hyperbolic fixed point or periodic orbit of a general Hamiltonian system with the same stability properties.

The Circular Restricted Three-Body Problem

The **Circular Restricted Three-Body Problem** describes the motion of a massless particle under the gravitational influence of two point masses m_1 and m_2 , called primaries, in circular motion around their common centre of mass. The equations of the CRTBP in an inertial reference frame can be obtained from the general equations of the Three Body Problem (Szebehely, 1967).

In a *synodic* reference system, which rotates with the same angular velocity as the primaries (both located on the $z = 0$ plane), and using a suitable set of units, such that the distance between the two primaries, the sum of their masses and the gravitational constant are all equal to one, the CRTBP can be written in Hamiltonian form with Hamiltonian function

$$H(x, y, z, p_z, p_y, p_z) = \frac{1}{2} \left(p_x^2 + p_y^2 + p_z^2 \right) - x p_y + y p_x - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2} \ ,$$

where $\mu = m_2/(m_1 + m_2) \in [0, 1]$ is the mass parameter, and r_1 and r_2 are the distances of the massless particle to m_1 (located at $(\mu - 1, 0, 0)$) and m_2 (located at $(\mu, 0, 0)$), respectively.
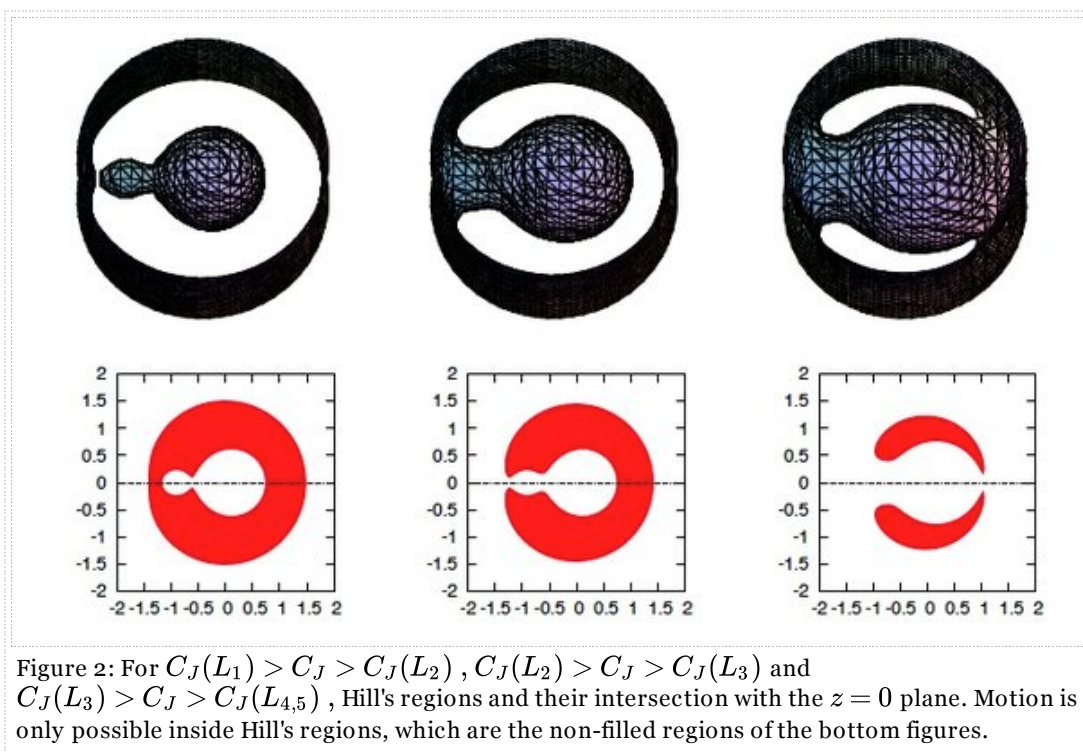
The constant value h of the Hamiltonian on each solution is usually referred as the *energy*, and it is related to the well known *Jacobi constant*, C_J by $C_J = -2h$. The *Jacobian integral* is usually written in synodic coordinates as

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = (x^2 + y^2) + 2 \frac{1 - \mu}{r_1} + 2 \frac{\mu}{r_2} - C_J \ . \tag{1}$$

The CRTBP has 5 equilibrium points, all of them in the plane of motion of the primaries, usually labeled L_i , $i = 1, 2, 3, 4, 5$. The first three points, also called collinear points, are on the x-axis

$$L_1$$

is located between the two primaries ($\mu - 1 < x_{L_1} < \mu$), L_2 is on the left hand side of both primaries ($x_{L_2} < \mu - 1$), and L_3 is on the right hand side of both primaries ($x_{L_3} > \mu$). The other two points, called triangular points, form equilateral triangles with the primaries.

Figure 2: For $C_J(L_1) > C_J > C_J(L_2)$, $C_J(L_2) > C_J > C_J(L_3)$ and $C_J(L_3) > C_J > C_J(L_{4,5})$, Hill's regions and their intersection with the $z = 0$ plane. Motion is only possible inside Hill's regions, which are the non-filled regions of the bottom figures.

Using Eq. (1), the surfaces of zero velocity are defined by $(x^2 + y^2) + 2\frac{1-\mu}{r_1} + 2\frac{\mu}{r_2} = C_J$ (see Figure 2). Their intersection with $z = 0$ define the zero-velocity curves. These surfaces define the boundary of the regions, in configuration space, for the admissible motions, and are known as Hill's regions. The topology of Hill's regions varies with the value of C_J and changes when C_J takes the values of the equilibrium points. For $C_J > C_J(L_1)$ the Hill's region has three components: two bounded, where the motion is confined around each primary, and an unbounded one outside the outermost zero velocity surface. At L_1 the two bounded components joint and and for $C_J < C_J(L_1)$ there is a bottleneck that connects the regions around both primaries. For $C_J < C_J(L_2)$ the equilibrium point L_2 appears, the Hill's region has only one unbounded component and the zero velocity curve on the $z = 0$ plane is horseshoe-shaped.

The phase space around the collinear equilibrium points: families of orbits and bifurcations
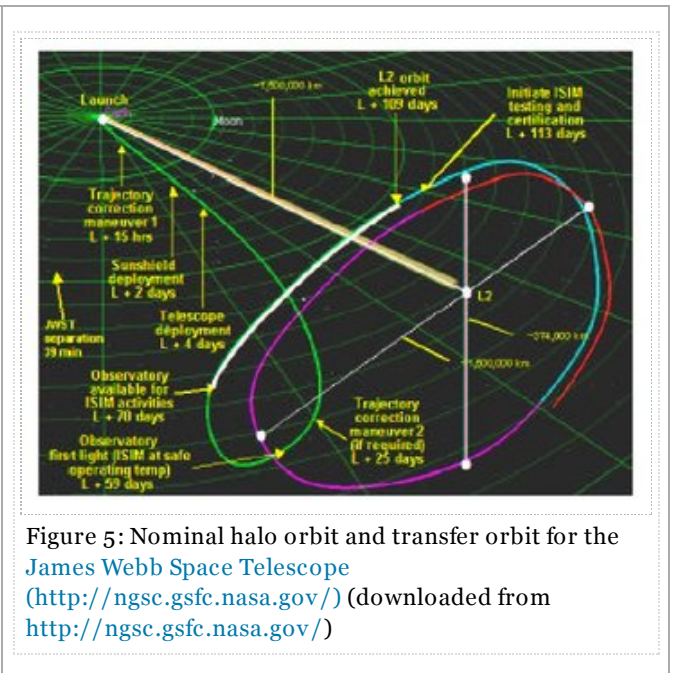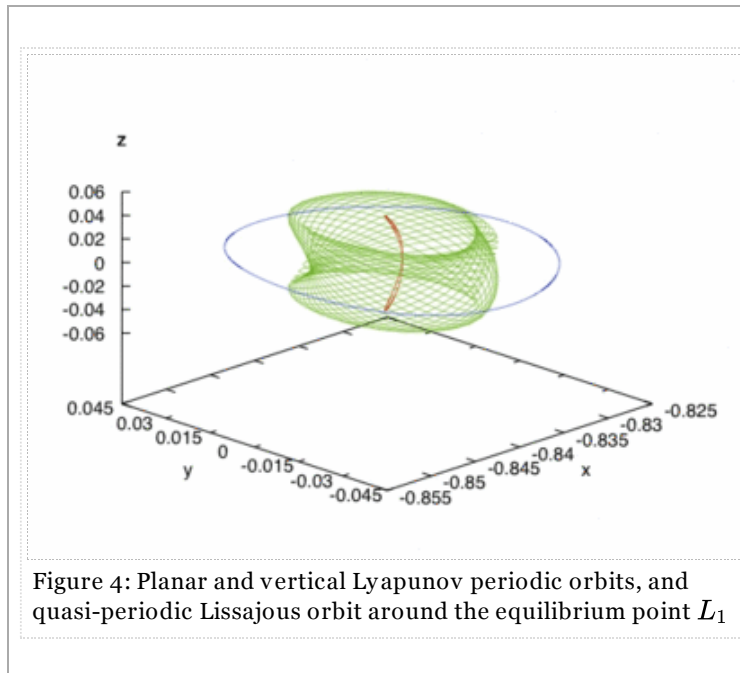
Since the eigenvalues of the Jacobian matrix of the CRTBP vector field at the three collinear equilibrium points are $\{\pm i\omega_1, \pm i\omega_2, \pm \lambda\}$, with $\omega_1, \omega_2, \lambda > 0$ and $i = \sqrt{-1}$, these points are of **centre** $\times$ **centre** $\times$ **saddle** type (Wiggins, 2003).

Figure 3: Behavior around the equilibrium points $L_i, i = 1, 2, 3$
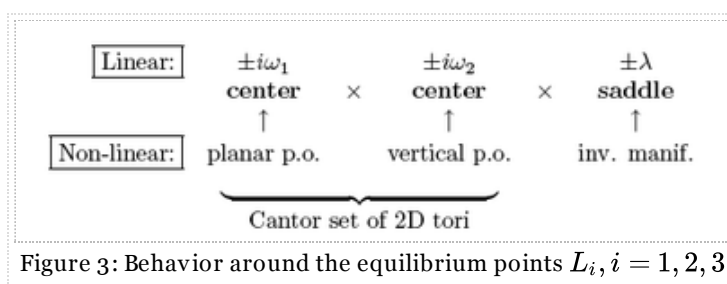
Due to the **centre** $\times$ **centre** part, and according to Lyapunov's centre theorem (Meyer and Hall, 1992), each collinear equilibrium point gives rise to two one-parameter families of periodic orbits, known as the planar and the vertical Lyapunov families of periodic orbits. In addition, in each energy level close to the one of the equilibrium point, there is a two-parameter family of 2D tori, known as *Lissajous orbits*, that connects the two Lyapunov families (see Figure 4 and Figure 6). Some of these tori are foliated by periodic orbits, but most of them carry an irrational flow. Thus, considering all the energy levels, there are 4D centre (neutrally stable) manifolds around these points. For a given energy level, they are just 3D sets where the dynamics has a *neutral behavior*.

Along the families of Lyapunov periodic orbits, as the energy h increases, the linear stability of the orbits change and there appear bifurcating orbits where other families of periodic orbits appear (see Figure 6). The first family bifurcating from the planar Lyapunov one corresponds to 3-dimensional periodic orbits symmetric with respect the $y = 0$ plane, the so-called **halo**

orbits (Farquar, 1968). At the bifurcation, two families of orbits are born, known as the Northern and Southern halo families. The name comes from the fact that the $y - z$ projections of such orbits in the Earth-Sun system as seen from the Earth look like a halo around the Sun. Halo orbits around the L_1 point of the Sun-Earth system were used in the ISEE-3 and SOHO missions.

Figure 4: Planar and vertical Lyapunov periodic orbits, and quasi-periodic Lissajous orbit around the equilibrium point L_1

Figure 5: Nominal halo orbit and transfer orbit for the James Webb Space Telescope (http://ngsc.gsfc.nasa.gov/) (downloaded from http://ngsc.gsfc.nasa.gov/)

All the objects in the centre manifold inherit the instability of the corresponding collinear equilibrium point, so all of them have associated stable/unstable manifolds, with an increase of one unit in their dimensions with respect to the orbit to which they are related.

The centre manifold of the collinear equilibrium points

The saddle component of the flow (associated to the eigenvalue $\lambda > 1$) makes the dynamics around the libration points that of an unstable equilibrium. In particular, in the case of L_1 and L_2 the eigenvalue λ is large and the instability is quite strong. Due to this fact, it is not possible to get an idea of the phase space in the vicinity of these points, as the one that could be given by a Poincaré map representation using direct numerical simulations. The instability must be removed in some way. To this purpose, two approaches can be done: to perform the reduction to the centre manifold, or to use semi-analytical (Lindstedt-Poincaré) or numerical methods to explicitly compute the orbits (periodic or quasi-periodic) of the centre manifold.

The reduction to the centre manifold

The reduction to the centre manifold, which is similar to a normal form computation, can be done in a semi-analytical way through a series of changes of variables which can be implemented by means of the Lie series method (Deprit, 1969). It consists in performing a reduction of the Hamiltonian that removes the hyperbolic directions, decreases the number of degrees of freedom, and allows the numerical study of the Poincaré map for the reduced Hamiltonian in a vicinity of the equilibrium points. Generically, the expansions required for these computations cannot be convergent in any open set, because of the crossing of resonances (Jorba and Masdemont, 1999).

The Hamiltonian of the CRTBP, with the second order terms in normal form, can be written in a suitable set of coordinates q_i and momenta p_i, as

$$H(q,p) = \sqrt{-1}\,\omega_1 q_1 p_1 + \sqrt{-1}\,\omega_2 q_2 p_2 + \lambda q_3 p_3 + \sum_{n \geq 3} H_n(q,p),$$

where H_n denotes an homogeneous polynomial of degree n.

To remove the instability associated with the hyperbolic character of H, note that the second order part of the Hamiltonian, H_2, displays the instability associated with the term $\lambda q_3 p_3$. In the linear approximation of the equations of motion, the central part is obtained setting $q_3 = p_3 = 0$. In order that a trajectory remains tangent to this space when adding the nonlinear terms, this is, with $q_3(t) = p_3(t) = 0$ for all $t > 0$, once setting $q_3(0) = p_3(0) = 0$, we need to have $\dot{q}_3(0) = \dot{p}_3(0) = 0$. Then, because of the autonomous character of the system, $q_3(t) = p_3(t) = 0$ for all $t \geq 0$.

Recalling the form of the Hamiltonian equations of motion, $\dot{q}_i = H_{p_i}$, $\dot{p}_i = -H_{q_i}$, the required condition, $\dot{q}_3(0) = \dot{p}_3(0) = 0$ when $q_3(0) = p_3(0) = 0$ will be fulfilled, if in the series expansion of the Hamiltonian, H, all the monomials, $h_{ij}q^i p^j$, with $i_3 + j_3 = 1$ have $h_{ij} = 0$ (i and j stand for (i_1, i_2, i_3) and (j_1, j_2, j_3), respectively). This happens if there are no monomials with $i_3 + j_3 = 1$.

All the computations of the reduction to the centre manifold can be implemented using specific symbolic manipulators that can carry out the full procedure up to an arbitrary order. In this way, we end up with a Hamiltonian $H(q, p) = H_N(q, p) + R_N(q, p)$, where $H_N(q, p)$ is a polynomial of degree N in (q, p) without terms with $i_3 + j_3 = 1$ and $R_N(q, p)$ is a remainder of order $N + 1$.

The Lindstedt-Poincaré method

The Lindstedt-Poincaré approach, for the explicit computation of central orbits, is some kind of collocation method that looks for formal expressions for periodic orbits or invariant tori in terms of suitable amplitudes and phases (Richardson, 1980).

For instance, for the Lissajous orbits around the vertical Lyapunov periodic orbits, the Lindstedt-Poincaré method proceeds as follows. The solutions of the linear CRTBP equations of motion around a collinear libration point are

$$
\begin{aligned}
x(t) &= \alpha \cos(\omega_1 t + \phi_1), \\
y(t) &= \kappa\alpha \sin(\omega_1 t + \phi_1), \\
z(t) &= \beta \cos(\omega_2 t + \phi_2),
\end{aligned}
$$

where ω_1 and ω_2 are the planar and vertical frequencies and κ is a constant. The parameters α and β are the in-plane and out-of-plane amplitudes of the orbit and ϕ_1,

Figure 6: Poincaré map representation of the central orbits associated to L_1 in the Earth-Sun system for $h = -1.575$.

ϕ_2 are the phases. These linear solutions are already Lissajous trajectories. When we consider the nonlinear terms, we look for formal series solutions in powers of the amplitudes α and β of the type

$$
\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{i,j=1}^{\infty} \left(\sum_{|k|\leq i, |m|\leq j} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_{ijkm} \begin{Bmatrix} \cos \\ \sin \\ \cos \end{Bmatrix} (k\theta_1 + m\theta_2) \right) \alpha^i \beta^j,
$$

where $\theta_1 = \omega t + \phi_1$ and $\theta_2 = \nu t + \phi_2$. Asking to these expansions to be a solution of the equations of motion, due to the presence of nonlinear terms, the frequencies ω and ν cannot be kept equal to ω_1 and ω_2, and they must be expanded in powers of the amplitudes $\omega = \omega_1 + \sum_{i,j=1}^{\infty} \omega_{ij}\alpha^i\beta^j$, $\nu = \omega_2 + \sum_{i,j=1}^{\infty} \nu_{ij}\alpha^i\beta^j$. The goal is to compute the coefficients x_{ijkm}, y_{ijkm}, z_{ijkm}, ω_{ij}, and ν_{ij} recurrently up to a finite order $N = i + j$.

The numerical approach

The numerical procedures for the determination of periodic and quasi-periodic orbits are based in Newton's method for the determination of the Fourier coefficients associated to the periodic orbit or to some invariant curve of the torus. To deal with the instability of the solutions the use of a multiple shooting method is mandatory.

Using the above methodologies, the orbits in the centre manifold can be computed. In Figure 6, Figure 7, the $x - y$ coordinates of these orbits, at their intersections with $z = 0$, are shown. The exterior curve in each plot is the Lyapunov planar orbit with the prescribed energy. For energy values close to the one of the equilibrium point, the whole picture is formed by invariant curves surrounding the fixed point associated to the Lyapunov vertical periodic orbit. They are related to the intersections of the Lissajous type trajectories. The halo orbits appear after the first bifurcation of the Lyapunov planar family that can be detected for values of the energy greater or equal to $h = -1.575$. As the vertical periodic orbit, they are also surrounded by invariant tori, the so called quasi-halo orbits. Other bifurcations can also be detected in the Poincaré representations.
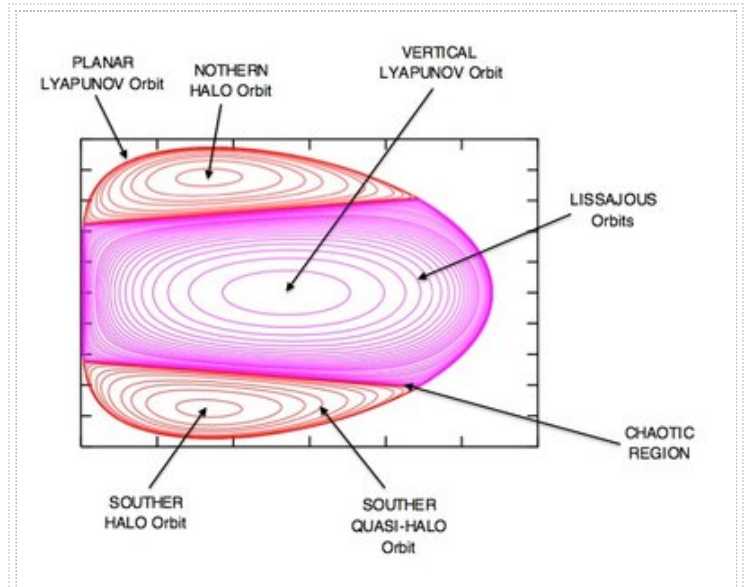
The region between the tori around the vertical Lyapunov orbit and the tori around the halo orbits is not empty, as it appears in the above figures, and should contain, at least, the traces on the surface of section of the invariant manifolds of the Lyapunov planar orbit. These manifolds act as separatrices between both kinds of motion. The same thing happens between the islands of the bifurcated halo-type orbits and the tori around the halo orbits. In this case, the region between both kinds of tori is filled with the traces of the invariant manifolds of the bifurcated hyperbolic halo-type orbits. In all these boundary regions, the motion should have a chaotic behavior.

Although the qualitative behavior between L_3 and $L_{1,2}$ is the same one, quantitatively they are different. The eigenvalue associated to the instability for L_3 is close to one, which is the responsible for the poor convergence radius of the reduction to the centre manifold. Hence, only a small range of values of the energy are inside the radius of convergence. In this case, directly numerical procedures for the computation of the Poincaré maps and the invariant manifolds have to be used.

Hyperbolic invariant manifolds

The hyperbolic collinear equilibrium points $L_i, i = 1, 2, 3$ have associated two 1-dimensional invariant manifolds, each one with two branches. As shown by Conley, 1968, in the linear approximation of the flow, a branch of the unstable manifold (W_-^u) departs to the $\{x < x_{L_i}, y > 0\}$ region, and the other (W_+^u) to the $\{x > x_{L_i}, y < 0\}$ region. By the symmetry of the CRTBP, the stable branches W_-^s, W_+^s, enter from the $\{x < x_{L_i}, y < 0\}, \{x > x_{L_i}, y > 0\}$ regions, respectively (see Figure 8).

Since the collinear equilibrium points L_i are hyperbolic, the orbits of the Lyapunov families, as well as the halo orbits, inherit the hyperbolicity and the periodic orbits have unstable and stable manifolds, W^u and W^s respectively, with a behavior similar to the one explained above. These manifolds are immersed 2-dimensional cylinders $\mathbb{R} \times S^1$. Geometrically, they can be viewed as two dimensional *tubes* approaching (forwards and backwards in time) towards a periodic orbit, that is, orbits *on* the invariant manifolds tend (forwards and backwards) to the periodic orbit.

If for a spacecraft mission an orbit of the central manifold (like a halo or Lissajous) is selected as nominal one, then the orbits of its stable manifold are the natural candidates to be used as transfer orbits.

Figure 7 : Poincaré map representation of the central manifold associated to L_1 for 4 different values of the energy (Gómez and Mondelo, 2001)

Some mission design applications

Transfers to libration point orbits

There are several approaches in the computation of transfer trajectories from the Earth to a libration point orbit. One uses direct shooting methods (forward or backward) together with a differential corrector, for
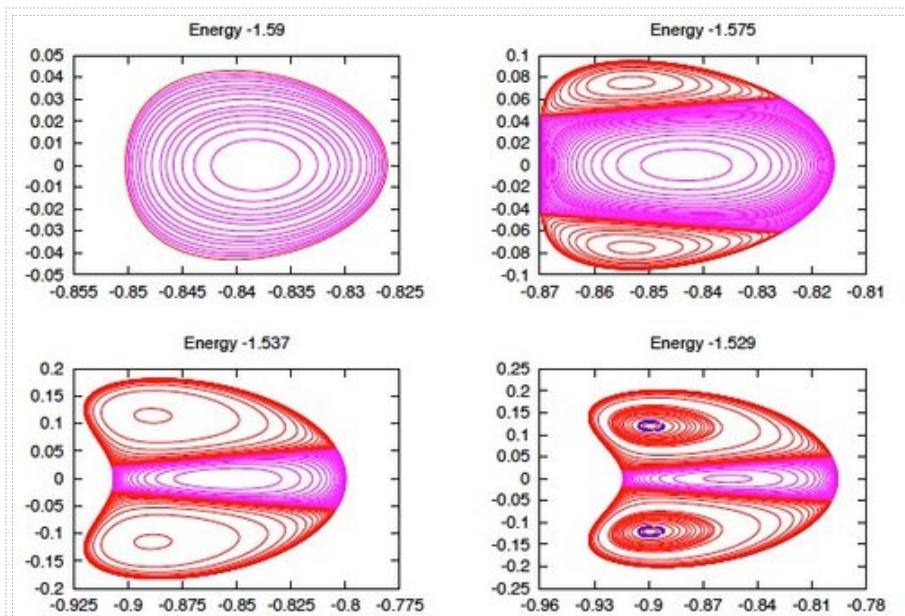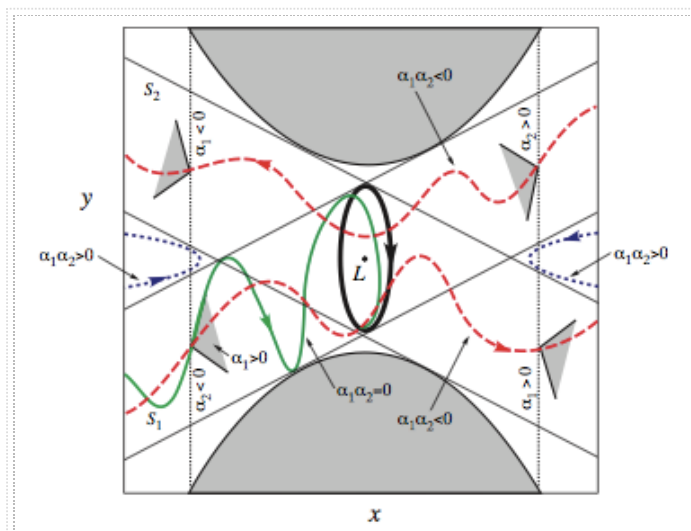
Figure 8: Sketch in configuration space of the dynamics in the bottleneck region around a unstable periodic orbit born from an hyperbolic equilibrium point. In blue, non-transit orbits, in red, transit orbits. The picture is a copy of the original one of Conley, 1968 obtained from Koon et al, 2006.

targeting and meeting mission goals. Proceeding in this way one can get direct or gravitational assisted transfers (Folta and Beckman, 2003).

Since the libration point orbits around L_1 and L_2 have a strong hyperbolic character, its is also possible to use their stable manifold for the transfer. This is the Dynamical Systems approach to the transfer problem (Gómez et al., 1993). It proceeds as follows:
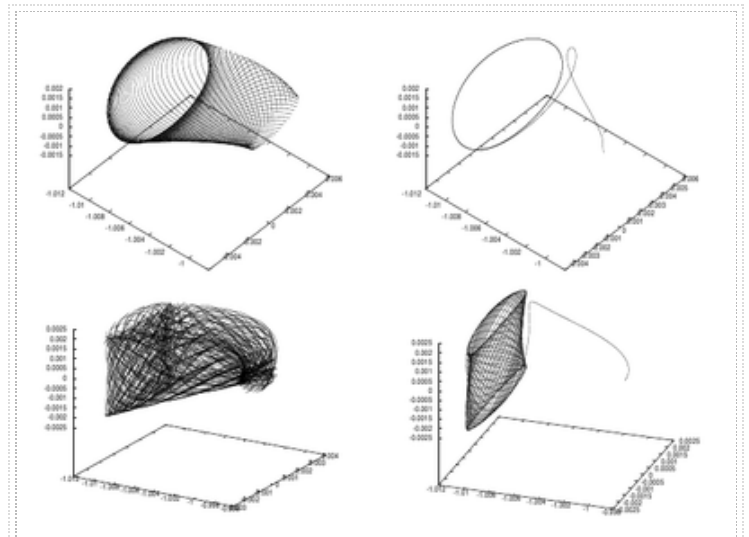

Figure 9: Illustration of the invariant manifold approach to the transfer to a halo orbit (top) and a Lissajous orbit (bottom). Using the orbits of the stable manifold (left hand side figures), the key idea of the transfer is to insert the satellite in an orbit having a close approach to the Earth (located near the point (1,0,0)) as the ones represented in the right hand side figures.

- Take a local approximation of the stable manifold at a certain number of points of the nominal orbit. At each base point, this determines a segment in the phase space.
- Propagate, backwards in time, the points in the segments of the local approximation of the stable manifold until one or several close approaches to the Earth are found (or until a maximum time span is reached). In this way some globalization of the stable manifold is obtained.
- Look at the possible intersections (in the configuration space) between the parking orbit of the spacecraft around the Earth and the stable manifold. At each one of these intersections the velocities in the stable manifold and in the parking orbit have different values, v_s and v_p . A perfect manoeuvre with $\Delta v = v_s - v_p$ will move the spacecraft from the parking orbit to an orbit that will reach the nominal orbit without any additional manoeuvre.
- Then Δv can be minimized by changing the base point of the nominal orbit at which the stable manifold has been computed (or any equivalent parameter).

Note that, depending on the nominal orbit and on the parking orbit, the intersection described in the third item can be empty, or the optimal solution found in this way can be too expensive. To overcome these difficulties several strategies can be adopted. One possibility is to perform a transfer to an orbit different from the nominal one and then, with some additional manoeuvres, move to the desired orbits. Another possibility is to allow for some intermediate manoeuvres in the path from the vicinity of the Earth to the final orbit.

As a matter of fact, in the Earth-Moon systems the hyperbolic manifolds associated with central orbits of L_1 and L_2 pass quite far from the Earth. To overcome this fact, one possible approach is a two manoeuvres transfer: one to depart from a Low Earth Orbit and one to insert either into the stable invariant manifold of the selected nominal orbit or even directly into the libration point orbit (Alessi et al.,2010).

The intersections between the invariant manifolds, giving rise to homoclinic or heteroclinic connections, also allow to construct itineraries that leave and finish in a neighborhood of two equilibrium points (the same one for the homoclinic connections, or a different one for the heteroclinic ones), see Figure 10. For example, the NASA's Artemis (http://www.nasa.gov/topics/solarsystem/sunearthsystem/main/News082510-artemis.html) spacecraft trajectory follows an heteroclinic connection between the two Lagrangian points (Figure 11). In this way, it is desirable to construct maps of heteroclinic and homoclinic connections between the invariant objects around the collinear equilibrium points. That maps can give a general idea of the characteristics of the different kind of possible connections (time of transfer, loops around the small primary, levels of energy, etc).
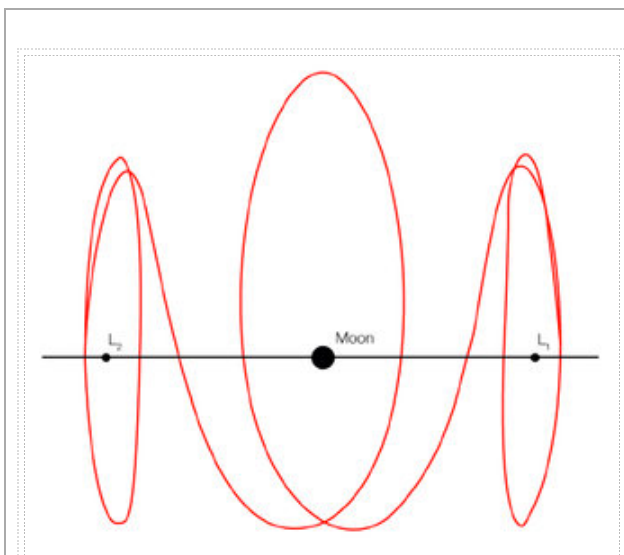
Figure 10: Heteroclinic connection between two planar Lyapunov orbits around the equilibrium points L_1 and L_2
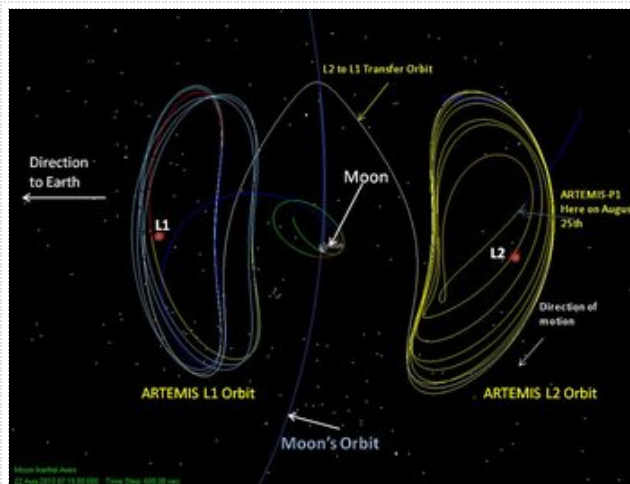

Figure 11: Trajectory of the Artemis (http://www.nasa.gov/topics/solarsystem/sunearthsyst artemis.html) spacecraft following a heteroclinic connection in the Earth-Moon system (downloaded from NASA's Artemis web page)

Transfers in a 4-body model: Shoot the Moon

After the work by Koon et al. 2001, it is well known that the so called low-energy transfers between the Earth and the Moon, as the ones determined by Miller and Belbruno, 1991 for the Muses-A/Hiten spacecraft, shadow orbits of the hyperbolic invariant manifolds of the Sun-Earth-spacecraft and Earth-Moon-spacecraft planar CRTBPs. Those hyperbolic manifolds
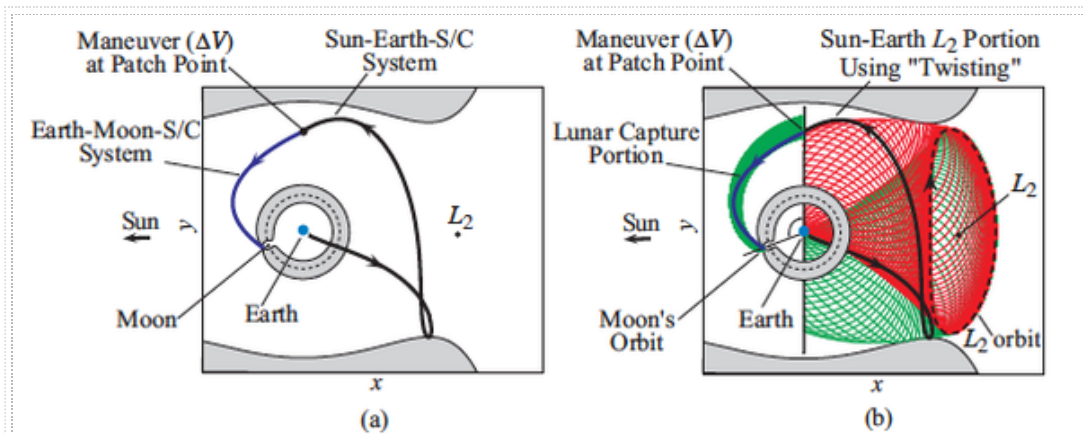

Figure 12: A low-energy transfer from the Earth to the Moon through the L_2 points of the Sun-Earth and Earth-Moon systems as seen in the Sun-Earth synodical barycentric reference frame (a), and its construction based on patching invariant manifolds of the Sun-Earth and Earth-Moon PCR3BPs (b). Figure taken from \cite{KLMR}.

are associated to orbits of the central manifolds of collinear libration points of the two restricted problems.

Because of this fact, it is possible to use the unstable manifolds of the planar Lyapunov periodic orbits about the Sun-Earth L_2 point to provide a low energy transfer from the Earth to the stable manifolds of planar Lyapunov periodic orbits around the Earth-Moon L_2 point. Furthermore, the stable manifolds of the periodic orbits around the Earth-Moon L_2 point, which act as separatrices in the energy manifold of the flow through the equilibrium point, provide the dynamical channels in phase space that enable ballistic captures of the spacecraft by the Moon (see Figure 12). Proceeding in this way one can compute in a systematic way low-energy Earth-Moon transfers using the combined gravitational influences of the Earth, the Moon and the Sun.

These ideas can also be applied to design low-energy transfers between two or more moons of Jupiter (JIMO (http://www.jpl.nasa.gov/jimo/) and Laplace/EJSM (http://sci.esa.int/science-e/www/area/index.cfm?fareaid=107) missions) or even to interplanetary transfers between the outer planets.

Station keeping at a libration point orbit

The basic purpose of station-keeping is to maintain a spacecraft within a predefined neighborhood about a nominal path. Roughly speaking, station-keeping is defined as a control procedure that seeks to minimize some combination of the deviation of the spacecraft from the nominal trajectory and the total manoeuvre cost, eventually subjected to constraints such as the

minimum time interval between two consecutive manoeuvre.

At a given epoch, the position and velocity of a spacecraft can only be estimated using tracking which, due to the errors, gives approximated values for both magnitudes. Due also to errors in the injection to the nominal orbit, in the execution of the control manoeuvres and in the model of the solar system used for the computation of the nominal trajectory, the spacecraft will never be exactly on the computed nominal orbit. All these facts have to be taken into account in the design of a station-keeping strategy.

For a given epoch, the error state is defined as the difference between the estimated actual state of the spacecraft and its nominal state, which can be defined as the point of the nominal orbit isochronous with the spacecraft. The evolution of an initial state error in time can be linearly approximated using the state transition matrix (solution of the variational equations). Most libration point orbits have six different Floquet modes: one is stable, so it is not necessary to take care about it, three are neutral and they do not increase even during large time intervals, one mode produces departures from the nominal orbit at a linear rate and finally the unstable one produces exponential departures. The component of the error along this last mode is multiplied by a factor equal to the corresponding eigenvalue (that in the typical examples
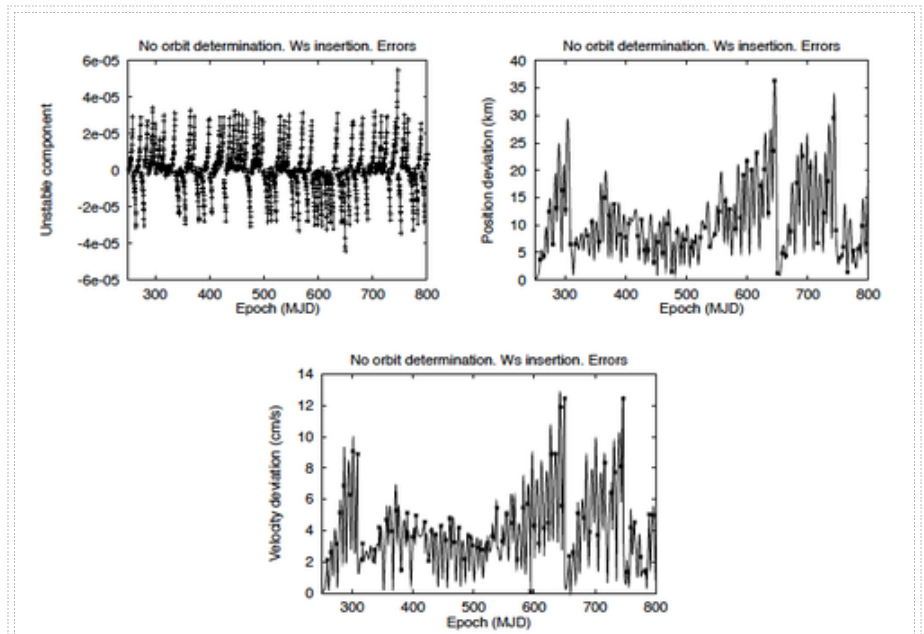

Figure 13: Evolution with time of the unstable component, position and velocity deviations with respect to the nominal trajectory using the station keeping strategy explained in the text.

is of the order of 10^3) at each revolution. The station keeping strategy must take care of this unstable mode (Simó et al., 1987). Figure 13 shows the evolution with time of the unstable component (together with position and velocity deviations) for a halo orbit around the L_1 point of the Sun-Earth system. In all the figures the manoeuvres are performed with errors. There are manoeuvres for the insertion in the stable manifold that can be clearly seen at the moments at which the distance to the nominal orbit decreases to very small values (which is not equal to zero because there is an error added to the manoeuvres). The points marked with a star are those at which a manoeuvre has been executed.

When the spacecraft departs from the nominal orbit, in principle it is not necessary to return to the nominal orbit again. The only thing that must be done is to cancel the unstable component of the deviation. However, in the real situation and for long time intervals, some modes that are neutral for the CRTBP, as well as the linearly unstable one, produce departures from the reference nominal orbit. Then, the cancellation of the unstable component will not be enough to perform an efficient station keeping and some other manoeuvres must be done. These manoeuvres will insert the spacecraft into the stable manifold of the nominal orbit.

In order to select how often manoeuvres have to be done, the departure of the spacecraft from the nominal orbit is the main parameter. The threshold in this magnitude must not be too small because the measured separation could only be due to tracking errors, moreover there will be very short time intervals between consecutive manoeuvres. On the other hand, it must not be too large because the cost of the manoeuvre increases in the same way as the separation between both trajectories. This is, exponentially. Finally, manoeuvres of insertion into the stable manifold will only be performed when the distance from the nominal orbit has become larger than a given amount. Again this amount must ensure the goodness of the different magnitudes used in the procedure which are computed by linear approximation. A typical evolution of the unstable component and the deviations with respect to the nominal trajectory is displayed in Figure 13.

Using this approach the station-keeping cost for a halo orbit, such as the one used by SOHO is of the order of 50 cm/s per year, which is about 4 times less that the Δv used by ISEE-3, for which the station keeping manoeuvres where designed to force the spacecraft to follow a certain nominal orbit. This is not only relevant because the total Δv decreases, but also because the

individual magnitude of the manoeuvres is much smaller. Too large manoeuvres produce vibrations on the spacecraft that don't allow to do some observations and measurements during relatively large time-intervals. This was the main reason why the station-keeping strategy used for SOHO was not the one used for ISEE3.

Orbits with prescribed itineraries

The dynamics around the equilibrium points L_i , for $i = 1, 2, 3$, is governed by the tubes of the invariant manifolds associated to the central manifold around them (Lyapunov periodic orbits, halo orbits and invariant tori), for the energies such that the invariant objects are hyperbolic. For such range of values, according the zero velocity surfaces, the configuration space can be divided in three regions: two around each primary, and a outer region far and around both. The two regions around the primaries are connected by a bottleneck around L_1 , while the region around the secondary and the exterior one are connected by a bottleneck around L_2 (see Figure 2). A particle can move from one region to another one following *transit orbits*, while it remains in the same region if it follows a *non-transit orbit*.

According to Conley, the tubes associated to the periodic orbits are the ones responsible of the transport between regions (see Figure 8). A particle inside a branch of the stable manifold will move from one region to another one via the bottleneck that connects both regions (it performs a transit), entering into the new region inside a branch of the unstable manifold. Depending whether this unstable branch intersects the stable one in the same region, the particle can follow a trajectory inside this stable branch and travel back again to the initial region (performing another transit). Instead, if the trajectory does not go inside the stable manifold anymore, it will remain in the same region (non-transit) .

Starting from the inner region around the big primary, a particle with an initial condition inside the tube of the stable branch $W_+^s(X_{L_1})$ (associate to a periodic orbit X_{L_1} around L_1) can move to the region around the small primary, following a trajectory inside the unstable branch $W_-^u(X_{L_1})$. Next, different scenarios can occur:

- The unstable branch $W_-^u(X_{L_1})$ intersects the stable branch $W_-^s(X_{L_1})$ (there exist a homoclinic connection) and the trajectory of the particle goes inside this stable tube: the trajectory will perform a transit back to the initial region.
- The unstable branch $W_-^u(X_{L_1})$ does not intersect the stable branch $W_-^s(X_{L_1})$ or it intersects but the trajectory of the particle does not go inside the stable tube. The trajectory will remain (for a while at least) into the region around m_2 .
- The unstable branch $W_-^u(X_{L_1})$ intersects the stable branch $W_+^s(X_{L_2})$ associated to a periodic orbit around L_2 with the same energy (there exist a heteroclinic connection) and the trajectory of the particle goes inside this stable tube: the trajectory will perform a transit towards the exterior region inside the unstable branch $W_-^u(X_{L_2})$.

Thus, depending whether the invariant manifolds intersect or not (thus, if there exist homo/heteroclinic connections), and the relative position of a particle with respect the tubes and their intersections, a trajectory can perform transits or non-transits between the regions of motion. Exploring the first intersections between the invariant manifolds via a Poincaré map defined on a suitable section, one can describe trajectories that stay around the big primary, travel to the small primary and eventually escape to the outer region (see Koon et al., 2006).

Iterating the process and studying the successive intersections of the branches of the invariant manifolds on a fixed and suitable sections, theoretically a symbolic dynamics can be defined such that each sequence of the dynamics correspond to a trajectory with a sequence of transits and non-transits. Thus, trajectories with prescribed itineraries of visits of the different regions can be designed just asking for initial conditions such that at each iteration of a suitable Poincaré map, the iterate is located in the corresponding region with respect transits and non-transits. In practice, numerically this is a delicate job: although the firsts intersections between the invariant manifolds can be easily computed, and the regions corresponding to transit and non-transit orbits can be identified, as the number of intersections increases, these regions become so narrow that are not anymore easy to identify.

Other applications

The knowledge of the phase space in the vicinity of the collinear libration points has allowed the design of other mission design applications. Some of them have to do with the transfer between libration point orbits around the same equilibrium point or between an orbit around L_1 and another around L_2 .

In relation with the first problem, consider the transfer between two halo orbits, H_1 and H_2 , around the same equilibrium point. Assume, for instance, that at a given epoch, t_1 , a spacecraft is on a halo orbit, H_1 , and that at this point a manoeuvre, $\Delta v^{(1)}$, is performed to go away from the actual orbit. At $t = t_2 > t_1$, a second manoeuvre, $\Delta v^{(2)}$, is executed in order to get

into the stable manifold of a nearby halo orbit H_2. Denoting by $\Delta\beta$ the difference between the z-amplitudes of these two orbits, the purpose of an optimal transfer is to perform both manoeuvres in such a way that the performance function

$$\frac{\Delta\beta}{\|\Delta v^{(1)}\|_2 + \|\Delta v^{(2)}\|_2},$$

is maximum.

The numerical computations show that the optimal first manoeuvre must be executed at a point, in the configuration space, always very close to the $z = 0$ plane. Once t_1, is fixed, there are, usually, two values of t_2 at which the performance function has a local maximum. The difference between these two values of t_2 is almost constant and equal to 180°. That is, after the first manoeuvre has been done, the two optimal possibilities appear separated by a difference of $1/2$ of revolution.

The cost of the transfer using the optimal t_2 is almost constant and the variation around the mean value does not exceed the 5%.

It can be seen (Gómez et al., 1998) that, for transfers between halo orbits around the L_1 point of the Sun-Earth system, the cost of a unitary transfer (to increase or decrease in one unit the z-amplitude of a halo orbit) is of 756 m/s and the behavior with the z-amplitude is almost linear. In this way, the cost of the transfer between two halo orbits

Figure 14: (x, y) and (y, z) projections of the transfer trajectory starting at a departure orbit of z-amplitude $\beta = 0.25$ and arriving at a final one with z-amplitude $\beta = 0.08$.

with z-amplitudes $\beta = 0.25$ and 0.08 is $(0.25 - 0.08) \times 756$ m/s = 128.5 m/s. The transfer between these two halo orbits is shown in Figure 14, where the dotted points correspond to the epochs at which the manoeuvres have been done.
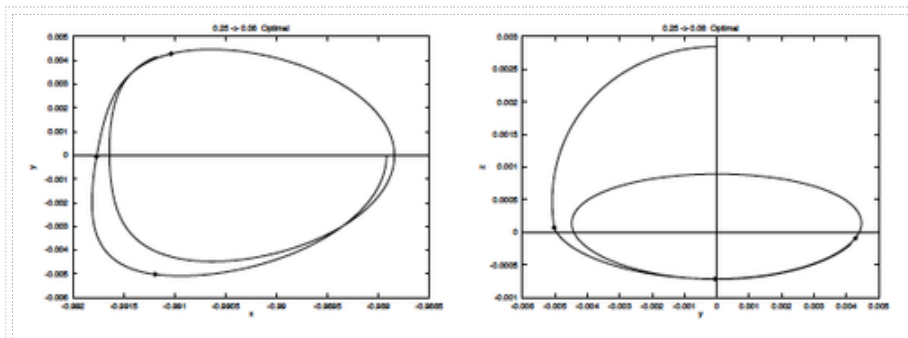
The study of the linearized CRTBP equations of motion about a collinear equilibrium point allows also the design of change of phase manoeuvres along a Lissajous orbit. These manoeuvres must be performed in order to avoid transits of the spacecraft, as seen from the Earth, in front of the solar disc (eclipse avoidance manoeuvres). The development was initiated during preliminary studies of the HERSCHEL/Plank mission and is fully developed in Cobos and Masdemont, 2003 and Canalias et al., 2003.

Further applications

Although the main attention has been focused in the collinear equilibrium points L_1 and L_2, the study of the dynamics around the collinear L_3 point has drawn some attention (Tantardini et al., 2010).

On one hand, the invariant manifolds associated to the central manifold around L_3 seem to play an important role in the effective stability regions around the triangular equilibrium points L_4 and L_5. On the other hand, we can find some astronomical applications. The dynamics around the collinear L_3 point allows to explain the horseshoe-type motion. This kind of motion has drawn some attention as it is performed by co-orbital satellites of Saturn, like Janus and Epimetheus, or by near Earth asteroids as Cruithne (http://en.wikipedia.org/wiki/3753_Cruithne) (an asteroid in 1:1 motion resonant with the Earth) or asteroid 2002 AA29 (http://en.wikipedia.org/wiki/2002_AA29). It is seen that the manifolds of both L_3 and its corresponding family of Lyapunov periodic orbits play a role in this kind of motion (Connors et al., 2002, Barrabés et al., 2009).

One of the problems for the usage of the invariant manifolds for astronautical purposes to visit a neighborhood of L_3 in the Sun-Earth system, is the time necessary to reach the equilibrium point from the Earth (of order of tens or hundreds of years). One possibility to reduce the flight time consist in adding a low-thrust propulsion to the spacecraft. There are recent works that combine stable/unstable manifolds with low-thrust propulsion. In some of them, the "Shoot the Moon" concept is reviewed under the perspective of low-thrust, as well as the strategy to transfers to equilibrium point orbits when the stable manifold does not intersect the departure orbit (Mingotti et al.,2009).

Studies of the CRTBP, in particular, the Sun-Earth-Moon system, have stimulated an enormous number of advances in physics and mathematics, including space manifold or chaotic dynamics in the Solar System. These (and other) topics are not exclusive of the classical celestial mechanics. The mechanisms of the restricted problem, as well as the methodologies explained above for the study of the dynamics, can also be found in systems that can be expressed in Hamiltonian form and with hyperbolic behaviours. That is the case of a large variety of microscopy systems (i.e., quantum systems) when the classical mechanic is considered (usually as a previous step before considering the quantum mechanics). For example, there is a large bibliography on perturbations of the hydrogen atom whose dynamics is described by the same (classical) Hamiltonian as is Kepler's problem (Brunello et al., 1997). Similarly, the dynamics of the problem of the ionization of the Rydberg atom interacting with external (electric or magnetic) fields can be described by a Hamiltonian formulation. Another (already) classical problem that is usually attacked using the manifolds dynamics is the transition state theory in Chemistry, that deals with the dynamics around a hypersurface that (roughly speaking) divides the phase space in *products* and *reactants* (Jaffé et al., 2001). The construction of such a surface makes use of the existence of the so called Normally Hyperbolic Invariant Manifold (NHIM) around near equilibrium points of saddle $\times$ centre $\times \ldots \times$ centre type.

Recently, works on the subject of **solar sailing** have gained interest, where the methodologies developed around the classical space dynamics can be applied to design control strategies for the navigation (see Farrés and Jorba, 2010). The basic model is the restricted three-body problem plus the effect of the solar radiation. In that case, a 2-dimensional family of equilibrium points appear, that allows to construct trajectories that can be kept around the equilibrium points for a long time, although their instability.

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See also

Dynamical systems, Hamiltonian systems, Equilibrium point, Three-body problem, Invariant manifold

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