# Group preference structures in AHP–group decision making

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**Abstract**

This paper presents a procedure that allows us to determine the preference structures *(PS)* associated to each of the different groups of actors that can be identified in a group decision making problem with a large number of individuals. To that end, it makes use of the *Analytic Hierarchy Process (AHP)* (Saaty, 1980) as the technique to solve discrete multicriteria decision making problems. This technique permits the resolution of multicriteria, multienvironment and multiactor problems in which subjective aspects and uncertainty have been incorporated into the model, constructing ratio scales corresponding to the priorities relative to the elements being compared, normalised in a distributive manner ($w_i = 1$). On the basis of the individuals' priorities we identify different clusters for the decision makers and, for each of these, the associated preference structure using, to that end, tools analogous to those of Multidimensional Scaling. The resulting *PS* will be employed to extract knowledge for the subsequent negotiation processes and, should it be necessary, to determine the relative importance of the alternatives being compared using anyone of the existing procedures.

**Key words:** Group Decision Making, AHP, Preference Structures, Cluster, Negotiation, Compositional Data.

## 1 Introduction

One of the most outstanding applications of group decision making in the social–economic environment is its use in public decision making, particularly into the field of *electronic democracy (e–democracy)*. It is in this context, and more specifically in the ambit of e–cognocracy, the new democratic system recently proposed by José María Moreno to extract and share knowledge relative to the scientific resolution of high complexity problems raised in the Social Sciences (Moreno–Jiménez, 2003; Moreno–Jiménez and Polasek, 2003), where the decisional tools developed by the consolidated research group of the Government of Aragón (Spain) known as "Grupo Decisión Multicriterio Zaragoza" (`http://gdmz.unizar.es`) acquire their real meaning.

Along these lines, we now present a new procedure to detect and identify patterns of behaviour in group decision making when the analytic hierarchy process *(AHP)* is used to prioritise and select between a discrete number of alternatives in a multiactor and multicriteria environment. This new methodological procedure based on a Bayesian analysis of the problem, will be carried out by developing new graphical visualization tools that favour: (i) the perception of consensus paths between the actors involved in the resolution of the problem; (ii) the interpretation of results and, finally, (iii) the extraction of the subjacent knowledge associated with the decisional process.

The paper has been structured as follows. Section 2 briefly presents the multicriteria decision making technique employed to solve the problem and introduces the importance of graphical visualization tools. Section 3 describes the algorithm proposed to identify the opinion groups and their characteristics (associated rankings). Section 4 illustrates the methodology by analysing a simple case study extracted from a real experiment on e–democracy developed for the city of Zaragoza (Spain). Section 5 closes the paper with a review of the main conclusions.

# 2 Background

## 2.1 The Analytic Hierarchy Process (AHP)

The Analytic Hierarchy Process is one of the most extended multicriteria decision making techniques. It was proposed by Thomas L. Saaty in the mid 1970s (Saaty, 1972, 1974, 1976, 1977, 1980), and combines tangible and intangible aspects to obtain, in a ratio scale, the priorities associated with the alternatives of the problem. Its methodology consists of four steps (Saaty, 1980): (1) Modelization, i.e. establishing a hierarchical representation of the problem, which should include all the relevant aspects of the decision problem; (2) Valuation, in which the decision maker incorporates his judgements through pairwise comparisons between the elements in the problem taken into consideration; (3) Priorization where the *local priorities* are obtained by using any of the existing priorization procedures (the eigenvector method –EGVM– and the row geometric mean method –RGMM– are the most widely used) and (4) Synthesis, in which the *total priorities* are derived. One of the main characteristics of AHP is the existence of a measure to evaluate the inconsistency of the decision maker when eliciting his judgements (Saaty, 1980; Aguarón and Moreno–Jiménez, 2003).

The complexity of the decisional problems that arise in the Social Sciences, together with the greater specialisation required of individuals in this new context, calls for the use of approaches more open and flexible than the traditional ones, and which have an appropriate behaviour in multiactor decision making (Saaty, 1996; Moreno–Jiménez *et al.*, 1999, 2002).

The consideration of multiple actors in high complexity social decision problems means that one of the desirable characteristics that we seek from the new decisional paradigms, along with publicity, rigour, simplicity and accessibility (Roy, 1993; Moreno–Jiménez *et al.*, 2001), is appropriate behaviour in group decision making (Saaty, 1994). Moreno–Jiménez *et al.* (2002) distinguish three different situations: (i) *Group Decision;* (ii) *Negotiated Decision* and (iii) *Systemic Decision.* In the first of these, all the individuals search for a common aim. In the second, each individual solves the problem independently, and agreement and disagreement zones among the different positions are sought for. Finally, in the third case, each individual acts independently, but all the positions are arrived at according to the tolerance principle.

In the following sections we focus on the identification of opinion groups through a Bayesian analysis of the problem and the development of new graphical visualization tools to favour the negotiation process in multiactor decision making.

## 2.2 Graphical Visualization Techniques

Over the last years a number of graphical representation techniques for multidimensional sets of data have been developed. These techniques try to help human minds to understand the world around us, that is to say, to explore this world, to interact with the environment and to extract knowledge that can be processed by our brain. Graphical visualization of information improves our response time and enables us to extract knowledge from this information more quickly. This result is especially true when the size or complexity of the data base complicates its analytical study. Furthermore, graphical visualization tools increase the degree of reliability in conclusions (Klein, 2002).

During the nineties and since the beginning of the $21^{st}$ century, the *Grupo Decisión Multicriterio Zaragoza (GDMZ)* has been developing different graphical visualization tools for the AHP. In this context, Turón *et al.* (2003) and Moreno–Jiménez *et al.* (2005) introduce graphics for the evolution of preference structures, the *value paths for PS and alternatives.* Turón and Moreno–Jiménez (2004) present the *density of judgment inconsistency diagram* and the *localization of PS diagram.* Finally, Turón *et al.* (2005) define the *consensus density diagram.* A more detailed study of general graphical visualization tools can be found in Asahi *et al.* (1994).

# 3 An algorithm for group identification

This section describes the algorithm used for finding the consensus groups from the set of decisors. The algorithm is a variation of the *k–media* algorithm, in which the role of the centroids is played by the consensus distribution of the group priorities vector, as defined in Gargallo *et al.* (2005).

## 3.1 Foundations

Assuming a local context (a single criterion), let $\mathbf{D} = \{D_1, \ldots, D_r\}, r \geq 2$ be a group of $r$ decision makers, emitting $r$ reciprocal pairwise comparison matrices $\{R^{(k)}_{n\times n}; k = 1, \ldots, r\}$ corresponding to the comparison, with respect to the considered criterion, of a set of $n$ decision alternatives $\{A_1, \ldots, A_n\}$, where $R^{(k)}_{n\times n} = \left(r^{(k)}_{ij}\right)$ is a square positive matrix verifying $R^{(k)}_{ii} = 1, r^{(k)}_{ji} = \frac{1}{r^{(k)}_{ij}} > 0$ for $i, j = 1, \ldots, n$. Judgements $r^{(k)}_{ij}$ represent the relative importance of alternative $i$ versus alternative $j$ for the decision maker $D_k$, following the fundamental scale proposed by Saaty (1980).

Let $\{\mathbf{v}^{(k)} = (v^{(k)}_1, \ldots, v^{(k)}_n)'; k = 1, \ldots, r\}$ $(v^{(k)}_1 > 0, \ldots, v^{(k)}_n > 0)$ be the local non–normalised priorities of the alternatives for each decision maker $D_k$, and let $\{\mathbf{w}^{(k)} = (w^{(k)}_1, \ldots, w^{(k)}_n)'; k = 1, \ldots, r\}$ be the same values normalised in a distributive manner $(\sum_{i=1}^n w^{(k)}_i = 1)$, that is, $w^{(k)}_i = \frac{v^{(k)}_i}{\sum_{i=1}^n v^{(k)}_i}$ for $i = 1, \ldots, n$.

The algorithm follows a Bayesian approach to the problem of group identification, and assumes that the judgement emitting process for each decision maker $D_k$ is given by a multiplicative model with log–normal errors, widely used in *AHP* literature:

$$r^{(k)}_{ij} = \frac{w^{(k)}_i}{w^{(k)}_j} e^{(k)}_{ij}; \quad i = 1, \ldots, n-1; \quad j = i+1, \ldots, n; \quad k = 1, \ldots, r. \tag{1}$$

Taking logarithms the following regression model with normal errors is obtained:

$$y^{(k)}_{ij} = \mu^{(k)}_i - \mu^{(k)}_j + \epsilon^{(k)}_{ij}; \quad i = 1, \ldots, n-1; \quad j = i+1, \ldots, n; \quad k = 1, \ldots, r \tag{2}$$

where $y^{(k)}_{ij} = \log(r^{(k)}_{ij})$, $\mu^{(k)}_i = \log(v^{(k)}_i)$ and $\epsilon^{(k)}_{ij} \sim N(0, {\sigma^{(k)}}^2)$ are independent for $k = 1, \ldots, r$. Moreover, with the aim of avoiding identificability problems, we take $\mu_n = 0$, that is, alternative $A_n$ is selected as reference alternative. As the prior distribution over $\mu^{(k)} = (\mu^{(k)}_1, \ldots, \mu^{(k)}_{n-1})'$ we take the uniform distribution on $\mathbb{R}^{n-1}$, which is the non–informative distribution usually taken in Bayesian literature.

Let $\mathbf{y}^{(k)} = (y^{(k)}_{12}, y^{(k)}_{13}, \ldots, y^{(k)}_{n-1\,n})'$ be the judgement vector for the decision maker $D_k$, $k = 1, \ldots, r$ and let $J = \frac{n(n-1)}{2}$ be the number of judgements emitted by that decisor. Let $\mathbf{X} = (x_{ij})$ be a matrix $J \times (n-1)$ so that if the $i-th$ component of vector $\{\mathbf{y}^{(k)}, k = 1, \ldots, r\}$ corresponds to the comparison between alternatives $A_j$ and $A_\ell$ with $1 \leq j < \ell < m$ then $x_{ij} = 1$, $x_{i\ell} = -1$ and $x_{is} = 0$ for $s \neq j, \ell$, and if it corresponds to a comparison between alternatives $A_j$ and $A_n$ with $1 \leq j < n$ then $x_{ij} = 1$ and $x_{is} = 0$ for $s \neq j$.

Eqs. (2) can be written, in matricial notation, as

$$\mathbf{y}^{(k)} = \mathbf{X}\boldsymbol{\mu}^{(k)} + \boldsymbol{\epsilon}^{(k)}; \quad k = 1, \ldots, r \tag{3}$$

with $\boldsymbol{\epsilon}^{(k)} = (\epsilon^{(k)}_{12}, \epsilon^{(k)}_{13}, \ldots, \epsilon^{(k)}_{n-1\,n})' \sim N_J(\mathbf{0}_J, {\sigma^{(k)}}^2\mathbf{I}_J)$.

The posterior distributions for $\{\boldsymbol{\mu}^{(k)}; k = 1, \ldots, r\}$ will be given by

$$\boldsymbol{\mu}^{(k)}|\mathbf{y}^{(k)} \sim N_{n-1}(\hat{\boldsymbol{\mu}}^{(k)}, {\sigma^{(k)}}^2(\mathbf{X}'\mathbf{X})^{-1}) \tag{4}$$

where $\hat{\boldsymbol{\mu}}^{(k)} = (\mathbf{X}'\mathbf{X}^{-1})(\mathbf{X}'\mathbf{y}^{(k)})$ is the *MLE* for $\boldsymbol{\mu}^{(k)}$.

## 3.2 Group consensus distribution

Let $\mathbf{v} = (v_1, \ldots, v_n)'$ and $\mathbf{w} = (w_1, \ldots, w_n)'$ be, respectively, the non–normalised and normalised priorities vectors for the group alternatives. Let $\mu = \log(\boldsymbol{v})$ be the log–priorities vector for the group.

Let $\{\beta_k;\ k = 1, \ldots, r\}$ ($\beta_1 > 0, \ldots, \beta_r > 0$; $\sum_{k=1}^{r} \beta_k = 1$) be the importance weights for each decision maker $D_1, \ldots, D_r$ in the group decision making process.

As shown in Gargallo *et al.* (2005), the consensus distribution for the group of decision makers is given by

$$\boldsymbol{\mu}|\mathbf{y} \sim N_{n-1}(\hat{\boldsymbol{\mu}}_{consensus}, \sigma^2_{consensus}(\mathbf{X}'\mathbf{X})^{-1}) \tag{5}$$

where

$$\hat{\boldsymbol{\mu}}_{consensus} = \frac{\sum_{k=1}^{r} \frac{\beta_k}{{\sigma^{(k)}}^2} \hat{\boldsymbol{\mu}}^{(\boldsymbol{k})}}{\sum_{k=1}^{r} \frac{\beta_k}{{\sigma^{(k)}}^2}} \quad \text{and} \quad \sigma^2_{consensus} = \frac{1}{\sum_{k=1}^{r} \frac{\beta_k}{{\sigma^{(k)}}^2}}$$

This distribution (Gargallo *et al.*, 2005) minimizes a weighted mean of the Kullback–Leibler distances from the group distribution to the posterior probability distributions.

By using this distribution, the consistency level of the group is estimated by the unbiased estimator traditionally used in *AHP* literature, given by

$$\hat{\sigma}^2_D = \frac{1}{J\,r - n + 1} \sum_{k=1}^{r} (\mathbf{y}^{(k)} - \mathbf{X}\hat{\boldsymbol{\mu}}_{consensus})'(\mathbf{y}^{(k)} - \mathbf{X}\hat{\boldsymbol{\mu}}_{consensus}) \tag{6}$$

This estimator is used for evaluating the level of consensus existing inside the group. To that purpose the upper consistency limit of the group, *lim*, is set in such a way that if the consistency level estimated for a group is greater than *lim*, there is not enough consensus in the group due to the great heterogeneity between the preferences of its members.

## 3.3 The general algorithm

Having established the statistical foundations of the algorithm, we now proceed to describe it in detail.

**Step 0 (Setting the upper level as a criterion for dividing or joining groups)**

The limit *lim* is set in order to determine which groups may be divided or joined.

**Step 1 (Initialization)**

***1a (Group construction)***

An initial partition of the set of decision makers $\mathbf{D}$, $\{G_1^{(0)}, \ldots, G_{k^{(0)}}^{(0)}\}$, is created. Set $k^{(0)} = n$ and

$$G_i^{(0)} = \{D_k \in \mathbf{D} : \hat{\mu}_i^{(k)} = \max_{1\le j\le n} \hat{\mu}_j^{(k)}\}; \quad i = 1,\ldots,n$$

where $\hat{\boldsymbol{\mu}}^{(k)} = (\hat{\mu}_1^{(k)}, \ldots, \hat{\mu}_{n-1}^{(k)})'$ and $\hat{\mu}_n^{(k)} = 0$. Remove empty groups.

***1b (Consensus distribution for every group)***

The consensus distribution of each group is calculated by using expression (5), so that $\boldsymbol{\mu}|G_i^{(0)} \sim N_{n-1}(\hat{\boldsymbol{\mu}}^{(i)}_{consensus}, {\sigma^{(i)}_{consensus}}^2(\mathbf{X}'\mathbf{X})^{-1})$, $i = i, \ldots, k^{(0)}$. Set $it = 1$ (number of iterations).

**Step 2 (Group assignment)**

For each decision maker $D_k$, the distance from its posterior distribution [Eq. (4)] to each consensus distribution of every one of the groups obtained in Step 1 is calculated. Each decisor is assigned to the nearest group. Once the process finishes empty groups are removed and a new partition $\{G_1^{(it)}, \ldots, G_{k^{(it)}}^{(it)}\}$ of the set of decision makers $\mathbf{D}$ is obtained.

**Step 3 (Consensus distribution for every group)**

The consensus distribution for every group is calculated by using Eq. (5), so that $\boldsymbol{\mu}|G_i^{(it)} \sim N_{n-1}(\hat{\boldsymbol{\mu}}^{(i)}_{consensus}, {\sigma^{(i)}_{consensus}}^2(\mathbf{X}'\mathbf{X})^{-1})$, $i = i, \ldots, k^{(it)}$.

**Step 4 (Group division)**

In order to analyze if any group has to be divided, the distance from the posterior distribution [Eq. (4)] of every decision maker to its group consensus distribution is obtained, after which the decision maker $D_k$ with the maximum distance is determined. If this distance is greater than $lim$ then the group $G_{i_{max}}^{(it)}$ to which the decision maker belongs is divided into two subgroups $\{D_k\}$ and $G_{i_{max}}^{(it)} - \{D_k\}$, and the consensus distribution of each group is calculated by using Eq. (5). Set $k^{(it+1)} = k^{(it)} + 1$, $it = it + 1$ and go to Step 3. Otherwise go to Step 6.

**Step 5 (Group joining)**

The pair $(i_{min}, j_{min}) \in C$, whose consensus distribution distance [Eq. (5)] is minimum, is obtained. If this distance is lower than $lim$, then $G_{i_{min}}^{(it)}$ and $G_{j_{min}}^{(it)}$ are replaced by $G_{i_{min}}^{(it)} \cup G_{j_{min}}^{(it)}$ and the consensus distribution for this new group is calculated. Set $k^{(it+1)} = k^{(it)} - 1$, $it = it + 1$ and go to Step 3. Otherwise go to Step 7.

**Step 6 (Ending condition)**

The algorithm finishes when two consecutive partitions of $\mathbf{D}$ are identical.

If the individual consistency levels $\{{\sigma^{(k)}}^2; k = 1, \ldots, r\}$ are unknown they can be estimated by means of the individual judgements emitted by each decision maker, using the unbiased estimator $\hat{\sigma}^{(k)^2} = \frac{1}{J-n+1}(\mathbf{y}^{(k)} - \mathbf{X}\hat{\boldsymbol{\mu}}^{(k)})'(\mathbf{y}^{(k)} - \mathbf{X}\hat{\boldsymbol{\mu}}^{(k)})$.

### 3.3.1 Classification algorithm for $P.\alpha$ problems

A $P.\alpha$ problem (Roy, 1993) aims to obtain the best alternative. This algorithm creates the groups depending on the similarities between the posterior distributions for the most preferred alternative in each group. It follows the outline of the general algorithm, using

$$D_\alpha(G,G') = \sqrt{w_\alpha(G)'w_\alpha(G) + w_\alpha(G')'w_\alpha(G') - 2w_\alpha(G)'w_\alpha(G')} \tag{7}$$

$G, G'$ being two subsets of the set of decision makers and $w_\alpha(G) = (w_1(G), \ldots, w_n(G))'$, with

$$w_i(G) = P\big(A_{(1)} = A_i | N_{n-1}(\boldsymbol{\mu}_{consensus}(G), \sigma^2_{consensus}(G)(\mathbf{X}'\mathbf{X})^{-1}\big); \; i = 1, \ldots, n \tag{8}$$

where $N_{n-1}(\boldsymbol{\mu}_{consensus}(G), \sigma^2_{consensus}(G)(\mathbf{X}'\mathbf{X})^{-1})$ is the consensus distribution [Eq. (5)] for group $G$.

### 3.3.2 Classification algorithm for $P.\gamma$ problems

A decisional problem is termed type $\gamma$ (Roy, 1993) if it aims to rank the alternatives. This algorithm constructs the groups depending on the similarities between the posterior distributions of the preference structures of each group. It follows the outline of the general algorithm, using the following distance between distributions:

$$D_\gamma(G,G') = 1 - \sqrt{w_\gamma(G)'\mathbf{T}w_\gamma(G) + w_\gamma(G')'\mathbf{T}w_\gamma(G') - 2w_\gamma(G)'\mathbf{T}w_\gamma(G')} \tag{9}$$

where $w_\gamma(G) = (w_s(G); \mathbf{s} \in S)$ for $S = \{i_1 > i_2 > \cdots > i_n; i_i \in \{1, \ldots, n\}; j = 1, \ldots, n\}$ set of preference structures,

$$w_s(G) = P\big(A_{(j)} = A_{i_j}; j = 1, \ldots, n | N_{n-1}(\boldsymbol{\mu}_{consensus}(G), \sigma^2_{consensus}(G)(\mathbf{X}'\mathbf{X})^{-1})\big); \; s \in S \tag{10}$$

$N_{n-1}(\boldsymbol{\mu}_{consensus}(G), \sigma^2_{consensus}(G)(\mathbf{X}'\mathbf{X})^{-1})$ being the consensus distribution [Eq. (5)] for group $G$ and $\mathbf{T}_{(n! \times n!)} = (\tau_{ss'})$, $s, s' \in S$, with $\tau_{ss'}$ the *tau* of Kendall between the preference structures $s$ and $s'$.

### 3.3.3 Classification algorithm based on consistency

This algorithm aims to create groups depending on the consistency level of the consensus priorities of each group with the judgements emitted by each one of its members. It uses, as the classification distance, the Kullback–Leibler distance from the posterior distributions [Eq. (1)] from each individual to each consensus distribution in the groups.

In order to determine the groups that will be joined or divided, an upper consistency limit *lim* is set. To that end, the one proposed by Altuzarra *et al.* (2005) is used:

$$P\left[\frac{1}{1+\frac{e_{max}}{100}} \leq \epsilon_{ij} \leq 1 + \frac{e_{max}}{100}\right] = 1 - \alpha \quad \text{for} \quad \epsilon_{ij} \sim LN(0, lim) \tag{11}$$

where $e_{max} > 0$ is the maximum error affordable in model (3) with a confidence level $100(1 - \alpha)\%$ $(0 < \alpha < 1)$.

To determine the group that will be divided an estimator of the inconsistency level $\hat{\sigma}^2_{G_i^{(it)}}$ of each group is calculated using Eq. (6). Then the set

$$G_{divide} = \left\{ i \in \{1, \ldots, k^{(it)}\} : \hat{\sigma}^2_{G_i^{(it)}} \geq lim \cdot \frac{\chi^2_{|G_i^{(it)}|J-n+1, 1-0.5\alpha}}{|G_i^{(it)}|J - n + 1}, \; |G_i^{(it)}| > 1 \right\} \tag{12}$$

is determined, the uper consistency thresholds being determined by means of the distribution $\chi^2$ as $\hat{\sigma}^2_{G_i^{(it)}}$ is an estimator of the consistency level of the group. If $G_{divide} = \varnothing$ go to Step 6. Otherwise the group $G^{(it)}_{i_{max}}$ such that $\hat{\sigma}^2_{G^{(it)}_{i_{max}}} = \max_{i \in G_{divide}} \hat{\sigma}^2_{G_i^{(it)}}$ is calculated. Let $D_k$ be de decision maker of $G^{(it)}_{i_{max}}$ such that ${\sigma^{(k)}}^2 = \max_{D_j \in G^{(it)}_{i_{max}}} {\sigma^{(j)}}^2$. $G^{(it)}_{i_{max}}$ is divided in two subgroups, $\{D_k\}$ and $G^{(it)}_{i_{max}} - \{D_k\}$, and the consensus distributions of every group are calculated using expression (5). Set $k^{(it+1)} = k^{(it)} + 1$, $it = it + 1$ and go to Step 3.

To establish which groups will be joined, the pair $(i_{min}, j_{min}) \in C$ such that $\hat{\sigma}^2_{G^{(it)}_{i_{min}} \cup G^{(it)}_{j_{min}}} = \min_{(i,j) \in C} \sigma^2_{G_i^{(it)} \cup G_j^{(it)}}$, is calculated, being

$$C = \Bigg\{ (i,j) \in \{1, \dots, k^{(it)}\} \times \{1, \dots, k^{(it)}\},\ 1 \le i < j \le k^{(it)}, \\ \hat{\sigma}^2_{G_i^{(it)} \cup G_j^{(it)}} \ge lim \cdot \frac{\chi^2_{|G_i^{(it)}|J-n+1, 1-0.5\alpha}}{|G_i^{(it)} \cup G_j^{(it)}|J - n + 1} \Bigg\} \tag{13}$$

If $C = \varnothing$ go to Step 7. Otherwise replace $G^{(it)}_{i_{min}}$ and $G^{(it)}_{j_{min}}$ by $G^{(it)}_{i_{min}} \cup G^{(it)}_{j_{min}}$ and calculate the consensus distribution of this group. Set $k^{(it+1)} = k^{(it)} - 1$, $it = it + 1$ and go to Step 3.

# 4 Example

In order to illustrate the proposed methodology, we have applied it to a simple case study extracted from a real experience of "e–participatory budget allocation" carried out by our research group *(GDMZ)* for the City Council of Zaragoza, Spain (see information in `http://www.zaragoza.es/presupuestosparticipativos/ElRabal/`).

Using *AHP* as the multicriteria methodological support and the Internet as the communication tool to obtain the preferences of each individual, the amount of the budget that the district of El Rabal (Zaragoza) assigns to each one of four alternatives proposed by the Neighbourhood Associations and the Members of the District Council has been obtained. The four alternatives have been prioritised taking into account three criteria and six subcriteria. In what follows, we consider exclusively the priorization problem presented at the first level of the hierarchy, where three criteria (Economic, Social and Environmental) hang from the goal of the problem.

From the individual priorities (see Appendix A) we obtain the opinion groups into which the decision makers are grouped and from these groups we obtain the PS associated with them. It is to these PS that we apply the graphical visualization tools (see figures on Appendix B).

## 4.1 Ternary diagrams

Ternary diagrams represent the individual priorities estimated by $\{\hat{\mathbf{w}}^{(k)} = (\hat{w}_1^{(k)}, \dots, \hat{w}_n^{(k)});\ k = 1, \dots, r\}$, where $\hat{w}_i^{(k)} = \frac{\exp(\hat{\mu}_i^{(k)})}{\sum_{j=1}^n \exp(\hat{\mu}_j^{(k)})}$; $i = 1, \dots, n$ and the consensus priorities of each group are estimated by $\hat{\mathbf{w}}_{consensus} = (w^{(i)}_{consensus,1}, \dots, w^{(i)}_{consensus,n})$, being $\hat{w}^{(i)}_{consensus,j} = \frac{\exp(\hat{\mu}^{(i)}_{consensus,j})}{\sum_{\ell=1}^n \exp(\hat{\mu}^{(k)}_{consensus,\ell})}$. The assigned groups are drawn in different colours and their consensus priorities are represented by circles. Each picture represents the results of applying the algorithm in Section 3.3 with different values for $lim$ (or $e_{max}$, in the case of the consistency levels, for which $\alpha = 0.05$ in the example). Notice that when the value of $lim$ increases (and therefore that of $e_{max}$, for the consistency levels), the number of groups is smaller. Moreover, the groups tend to be located in the areas of the diagram in which one of the alternatives is preferred to the other two, and inside the triangular zones corresponding to subzones in which the most preferred structures are located. This can be seen in Figs. 1 to 5. For example, in the case $e_{max} = 200$ (Fig. 5A) there are nine groups, the individuals being represented by asterisks and the group centroids by circles. This fact allows us to characterise the global behaviour of the group. In the case $e_{max} = 700$ (Fig. 5D) the number of groups has been reduced to two, so some individuals have been relocated in the nearest groups.

When observing the behaviour of some individuals we can conclude that the ternary diagram does not intuitively represent this "nearness", since when moving from $e_{max} = 200$ to $e_{max} = 700$ some of them are not reallocated in the nearest groups, if we consider the euclidean distance in the plane of the simplex. The three–dimensional graphics shown in Fig. 9 allow us to get a clearer vision of the nearness between individuals and groups.

In type $\alpha$ problems, the diagrams are constructed from the posterior probabilities $w_\alpha(G) = (w_1(G), \ldots, w_n(G))'$. In type $\gamma$ problems, they are constructed from the probabilities $w_\gamma(G) = \{w_s(G); \mathbf{s} \in S\}$. In this case we have also shown (Fig. 4) the priorities distribution with respect to the most preferred pair of structures versus the rest (*Others*).

## 4.2 Multidimensional scaling (MDS) diagrams

The *MDS* pictures corresponding to the most preferred alternatives (Fig. 6) have been obtained from the results of applying *Multidimensional Scaling* to the matrix of distances

$$D_\alpha = (d^\alpha_{k\ell})\,(Matrix(r + ngroups + n) \times (Matrix(r + ngroups + n)) \tag{14}$$

where $d^\alpha_{k\ell} = D_\alpha(G_k, G_\ell)$, $\{G_k;\, k = 1, \ldots, ngroups\}$ being either one of the groups found or one of the single element groups $\{D_k;\, k = 1, \ldots, r\}$, or one of the $n$ distributions degenerated in each of the problem alternatives.

The *MDS* pictures corresponding to the preference structures (Fig. 7) have been obtained by a similar procedure, applying *Multidimensional Scaling* to the matrix of distances

$$D_\gamma = (d^\gamma_{k\ell})\,(Matrix(r + ngroups + n!) \times (Matrix(r + ngroups + n!)) \tag{15}$$

where $d^\gamma_{k\ell} = D_\gamma(G_k, G_\ell)$, $\{G_k;\, k = 1, \ldots, ngroups\}$ being either one of the groups that have been found or one of the single element groups $\{D_k;\, k = 1, \ldots, r\}$, or one of the $n!$ distributions degenerated in the preference structures.

The *MDS* pictures corresponding to the consistency levels (Fig. 8) have been obtained by applying *Multidimensional Scaling* to the matrix of distances

$$D = (d_{k\ell})\,(Matrix(r + ngroups + n!) \times (Matriz(r + ngroups + n!)) \tag{16}$$

where

$$d_{k\ell} = \sum_{i=1}^{n!}\sum_{j=1}^{n!} p^{(k)}_{Ranking_i} p^{(\ell)}_{Ranking_j}\, Dist(Ranking_i, Ranking_j)$$

with $p^{(k)}_{Ranking} = P\lfloor Ranking|\boldsymbol{\mu}^{(k)} \sim N_{n-1}(\mathbf{m}^{(k)}, \sigma^{(k)}(\mathbf{X}'\mathbf{X})^{-1})\rfloor$ being the probability of the preference structure over the alternatives $\{A_1, \ldots, A_n\}$ obtained from the posterior distribution of the log–priorities vector $\boldsymbol{\mu}^{(k)}$

$$Dist(Ranking_i, Ranking_j) = 1 - \rho_{Ranking_i, Ranking_j} \quad (17)$$

$\rho_{Ranking_i, Ranking_j}$ being the Spearman rank correlation coefficient.

The points represented here correspond to the decision makers (represented by $*$) that use the posterior distributions [Eq. (4)], to the groups (represented by $\circ$) that use their consensus distributions [Eq. (5)], and the rankings (represented by traingles) that use their degenerated distributions. The label of each ranking represents the preference of the alternatives (*e.g.* 312 is preference structure $\{A_3, A_1, A_2\}$).

### 4.3 Consensus density diagrams

The *consensus density diagrams* represent the density of the spatial distribution of the different preference structures. For this purpose they use the consensus distribution expressed by Eq. (5) generating, by computer simulation, a set of distribution values and then obtaining the individual priorities, estimated as in 4.1. There has to be a sufficient number of values to allow the calculation of the density of priorities in every point of the simplex, so that the density distribution over it can be obtained. Such a distribution can be represented either by means of a three–dimensional picture or by means of a diagram in which every point is represented by a different colour intensity depending on its density.

In Fig. 9 pictures of both kinds are shown. As can be seen, the three–dimensional view gives wider information on the position of every decision maker with respect to his consensus group, whilst ternary diagrams do not show this information.

The three–dimensional graphics of Fig. 9 have been obtained from a geometric model of the simulated data; this fact allows us to modify the viewpoint interactively by means of a 3D visualization tool, which represents a notable aid in the phase of data interpretation and knowledge extraction, since this kind of tools make it easy to identify patterns associated with the different attitudes, either individual or collective, in the set of decision makers.

## 5 Conclusions

This paper presents a new classification algorithm that allows us to detect opinion groups in *AHP* group decision making, as well as to construct graphical visualization tools that capture the behaviour of preference structures and help human minds to understand problems and their possible consensus solutions. These tools include *ternary diagrams*, *multidimensional scaling (MDS) diagrams* and *consensus density diagrams*. From these tools different consensus paths between the actors involved in the resolution of the problem can be proposed.

Finally, it must be pointed out that a new three–dimensional graphic that incorporates the distributions of clusters represented by those of their centroids has been presented. This graph shows the density of the PS spatial distributions derived from the resolution of the problem, and allows us to detect the behaviour patterns that exist in the problem and to know the relative importance of the actors in the final consensus solution. The new graphics can be examined from different points of view in an interactive way by using appropriate visualization tools. Moreover, these graphics provide the relative position of the actors' priorities in a three–dimensional space.

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# Appendix A

**Table 1:** Judgements and priorities for the criteria

| DM | $\beta_k$ | $a_{12}$ | $a_{13}$ | $a_{23}$ | $w_1$ | $w_2$ | $w_3$ |
|---|---|---|---|---|---|---|---|
| 1 | 8.0 | −3 | −5 | −5 | 0.3914 | 0.2784 | 0.3301 |
| 2 | 8.0 | 5 | −5 | 7 | 0.6724 | 0.2573 | 0.0703 |
| 3 | 8.0 | 7 | −7 | 7 | 0.7419 | 0.2027 | 0.0554 |
| 4 | 8.0 | 5 | 5 | 5 | 0.3333 | 0.3333 | 0.3333 |
| 5 | 8.0 | 1 | −3 | 5 | 0.4054 | 0.4806 | 0.1140 |
| 6 | 0.6 | −9 | 5 | 1 | 0.0691 | 0.5109 | 0.4200 |
| 7 | 0.6 | −9 | 5 | 1 | 0.0691 | 0.5109 | 0.4200 |
| 8 | 0.6 | −9 | 5 | 1 | 0.0691 | 0.5109 | 0.4200 |
| 9 | 0.6 | −9 | 7 | 1 | 0.0592 | 0.4901 | 0.4507 |
| 10 | 0.6 | −9 | 7 | 1 | 0.0592 | 0.4901 | 0.4507 |
| 11 | 0.6 | −9 | 5 | 1 | 0.0691 | 0.5109 | 0.4200 |
| 12 | 0.6 | −9 | 5 | 1 | 0.0691 | 0.5109 | 0.4200 |
| 13 | 0.6 | −9 | 9 | 9 | 0.0416 | 0.7785 | 0.1799 |
| 14 | 0.6 | −9 | 9 | 9 | 0.0416 | 0.7785 | 0.1799 |
| 15 | 0.6 | −9 | 5 | 1 | 0.0691 | 0.5109 | 0.4200 |
| 16 | 0.6 | −9 | 5 | 1 | 0.0691 | 0.5109 | 0.4200 |
| 17 | 0.6 | −9 | 5 | 1 | 0.0691 | 0.5109 | 0.4200 |
| 18 | 0.6 | −9 | 5 | 1 | 0.0691 | 0.5109 | 0.4200 |
| 19 | 0.6 | −9 | 5 | 1 | 0.0691 | 0.5109 | 0.4200 |
| 20 | 0.6 | −9 | 5 | 1 | 0.0691 | 0.5109 | 0.4200 |
| 21 | 0.6 | −9 | 5 | 1 | 0.0691 | 0.5109 | 0.4200 |
| 22 | 0.6 | −9 | 5 | 1 | 0.0691 | 0.5109 | 0.4200 |
| 23 | 0.6 | −9 | 5 | 1 | 0.0691 | 0.5109 | 0.4200 |
| 24 | 0.6 | −9 | 5 | 1 | 0.0691 | 0.5109 | 0.4200 |
| 25 | 0.6 | −9 | 5 | 1 | 0.0691 | 0.5109 | 0.4200 |
| 26 | 2.4 | −5 | 9 | −7 | 0.0545 | 0.1734 | 0.7720 |
| 27 | 2.4 | −7 | 7 | −5 | 0.0586 | 0.2399 | 0.7015 |
| 28 | 2.4 | −7 | 5 | −5 | 0.0703 | 0.2573 | 0.6724 |
| 29 | 2.4 | −5 | −5 | 1 | 0.3035 | 0.5190 | 0.1775 |
| 30 | 2.4 | −5 | 5 | 1 | 0.0909 | 0.4545 | 0.4545 |
| 31 | 6.0 | −9 | 7 | 5 | 0.0510 | 0.7219 | 0.2271 |
| 32 | 6.0 | −7 | 5 | 9 | 0.0599 | 0.7792 | 0.1610 |
| 33 | 6.0 | −3 | −3 | 1 | 0.3189 | 0.4600 | 0.2211 |
| 34 | 2.0 | −9 | 9 | 1 | 0.0526 | 0.4737 | 0.4737 |
| 35 | 3.0 | −5 | 7 | 5 | 0.0703 | 0.6724 | 0.2573 |
| 36 | 3.0 | 5 | 5 | −5 | 0.2344 | 0.0802 | 0.6854 |

# Appendix B

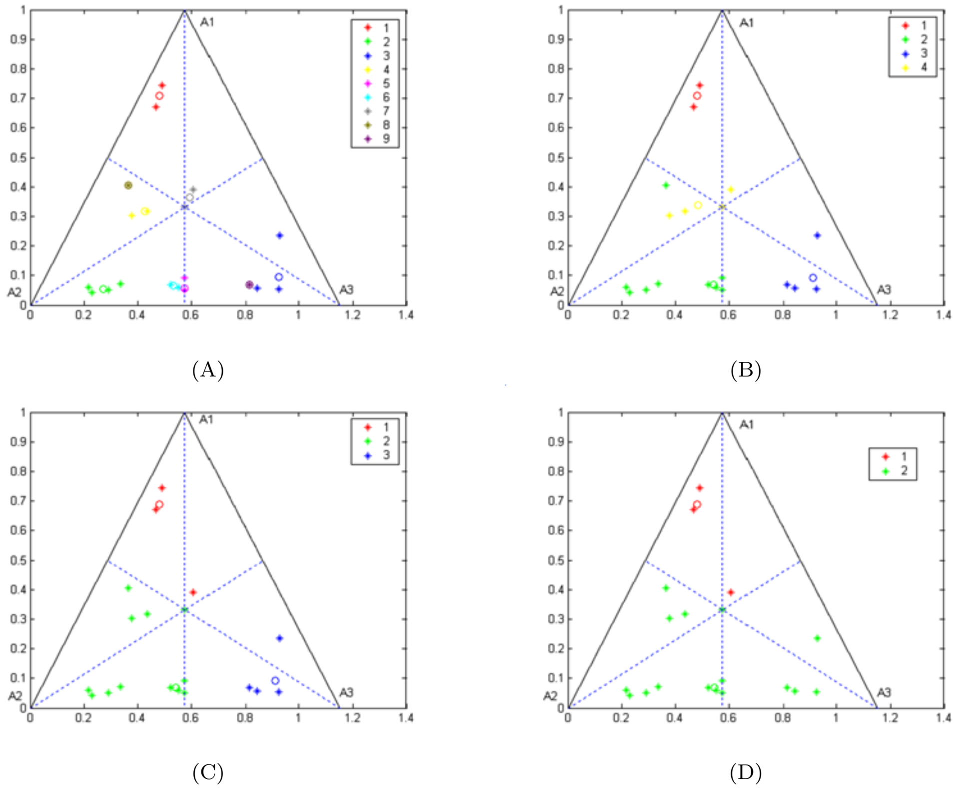
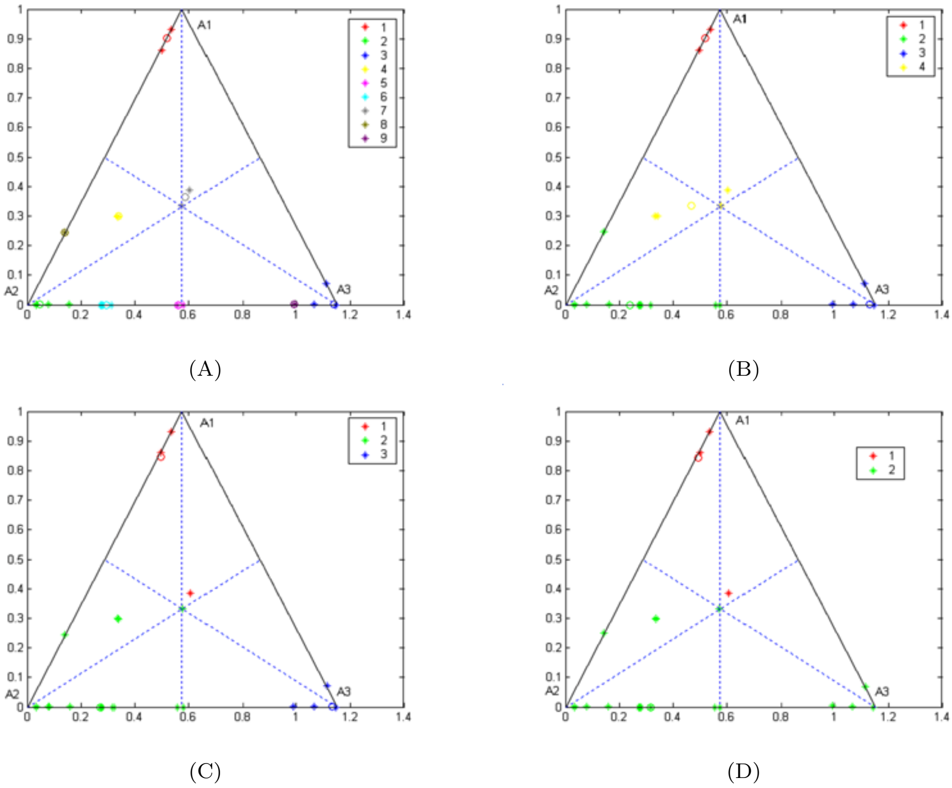



**Figure** 1: Ternary diagrams with the priorities for the most preferred alternatives ($\alpha$): (A) $lim =$ 0.00; (B) $lim = 0.25$ (C) $lim = 0.50$; (D) $lim = 0.75$.




(A) (B) (C) (D)

**Figure** 2: Ternary diagrams for the most preferred alternatives ($\alpha$): (A) $lim = 0.00$; (B) $lim = 0.25$ (C) $lim = 0.50$; (D) $lim = 0.75$.

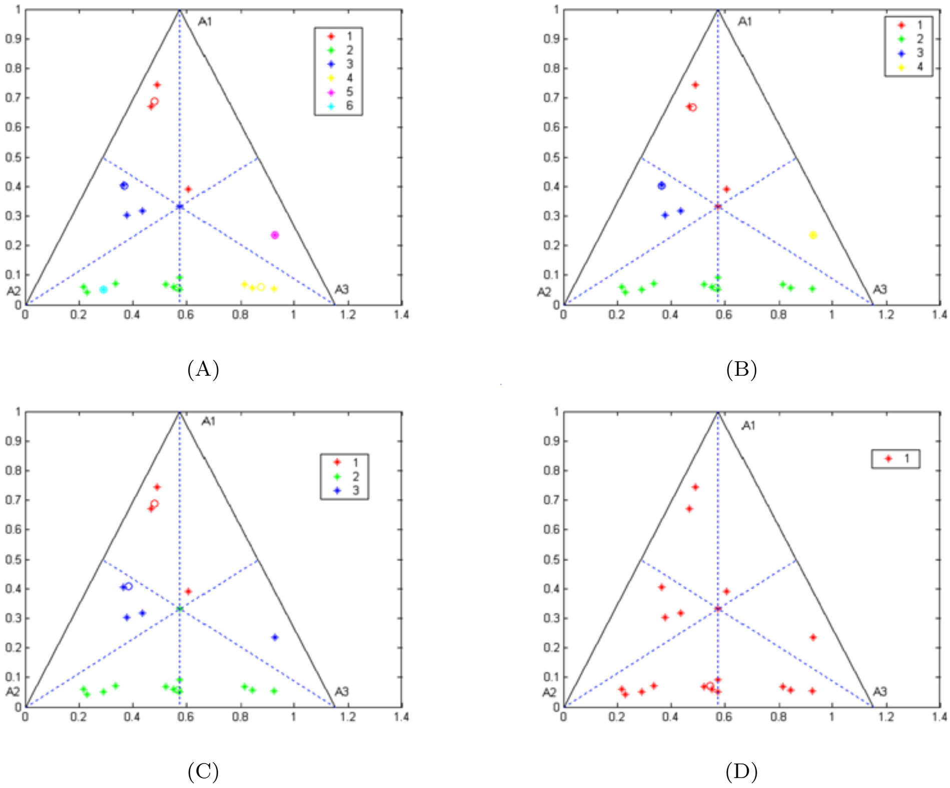



**Figure** 3: Ternary diagrams with the priorities for the most preference structures ($\gamma$): (A) $lim = 0.00$; (B) $lim = 0.25$ (C) $lim = 1.00$; (D) $lim = 2.00$.

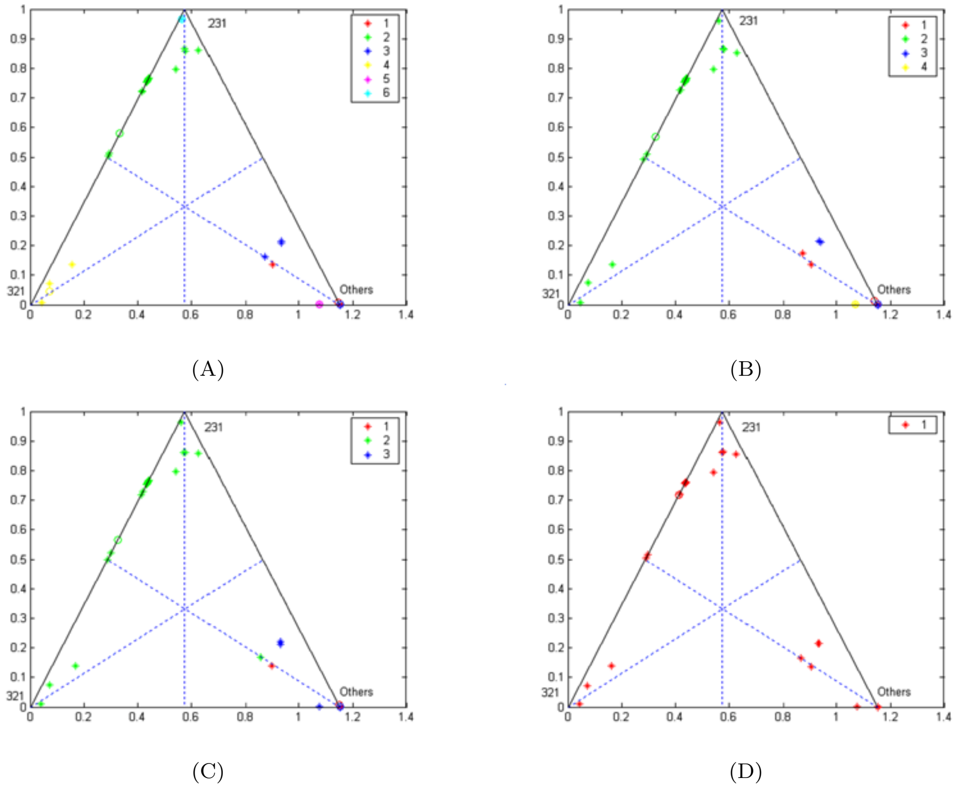



**Figure** 4: Ternary diagrams for the preference structures ($\gamma$): (A) $lim = 0.00$; (B) $lim = 0.25$ (C) $lim = 1.00$; (D) $lim = 2.00$.

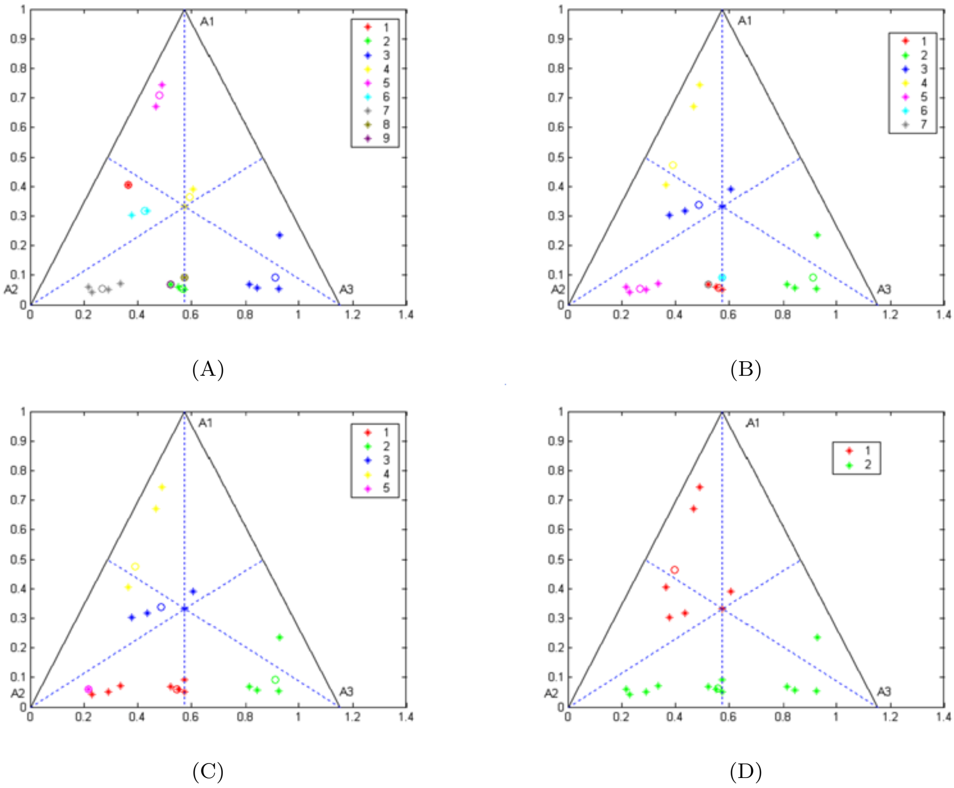



**Figure** 5: Ternary diagrams with the priorities for the consistency levels: (A) $e_{max} = 200$; (B) $e_{max} = 600$ (C) $e_{max} = 650$; (D) $e_{max} = 700$.

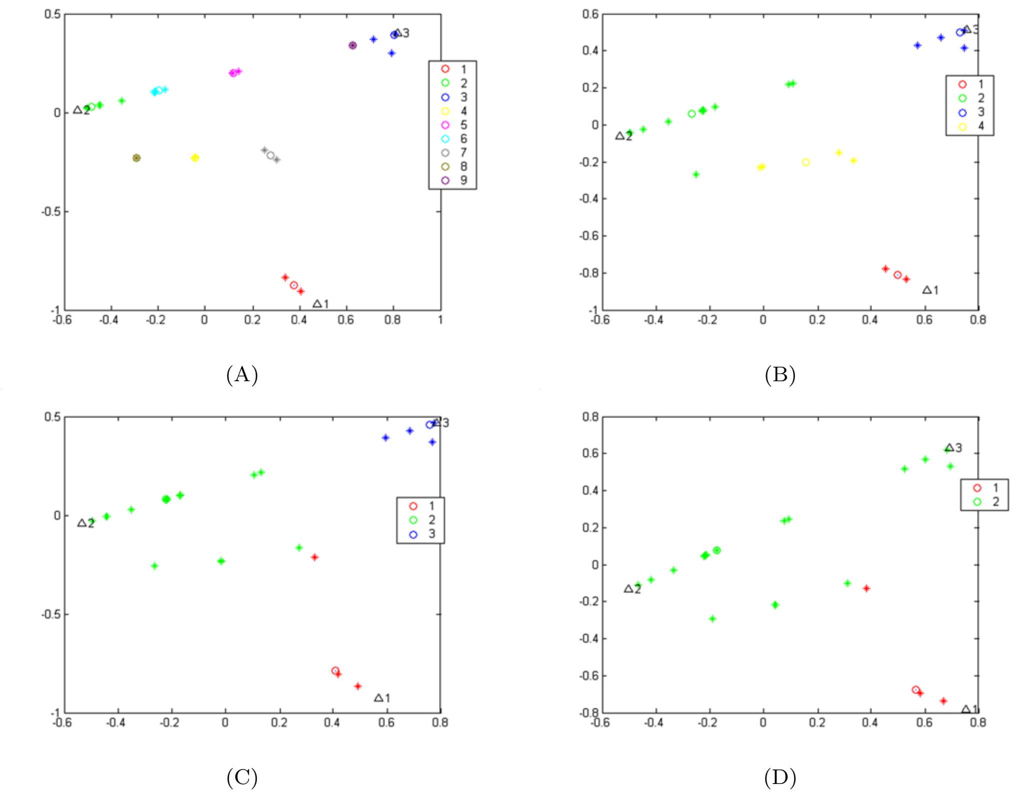



**Figure** 6: *MDS* diagrams for the most preferred alternatives ($\alpha$): (A) $lim = 0.00$; (B) $lim = 0.25$ (C) $lim = 0.50$; (D) $lim = 0.75$.

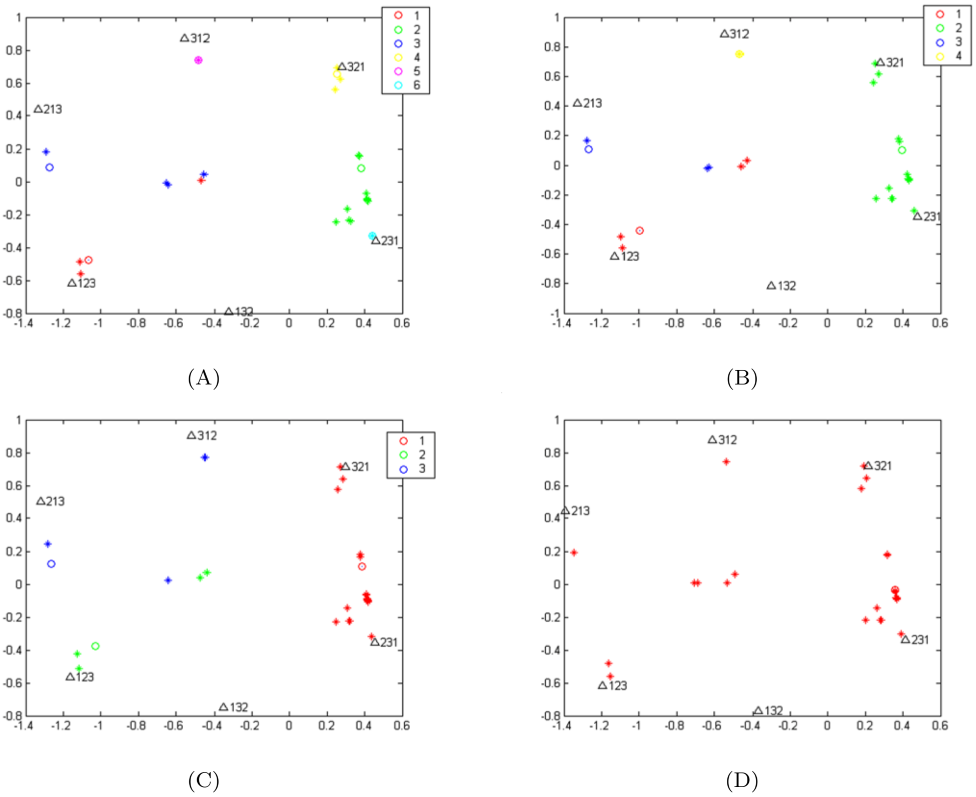



**Figure 7**: $MDS$ diagrams for the preference structures ($\gamma$): (A) $lim = 0.00$; (B) $lim = 0.25$ (C) $lim = 1.00$; (D) $lim = 2.00$.

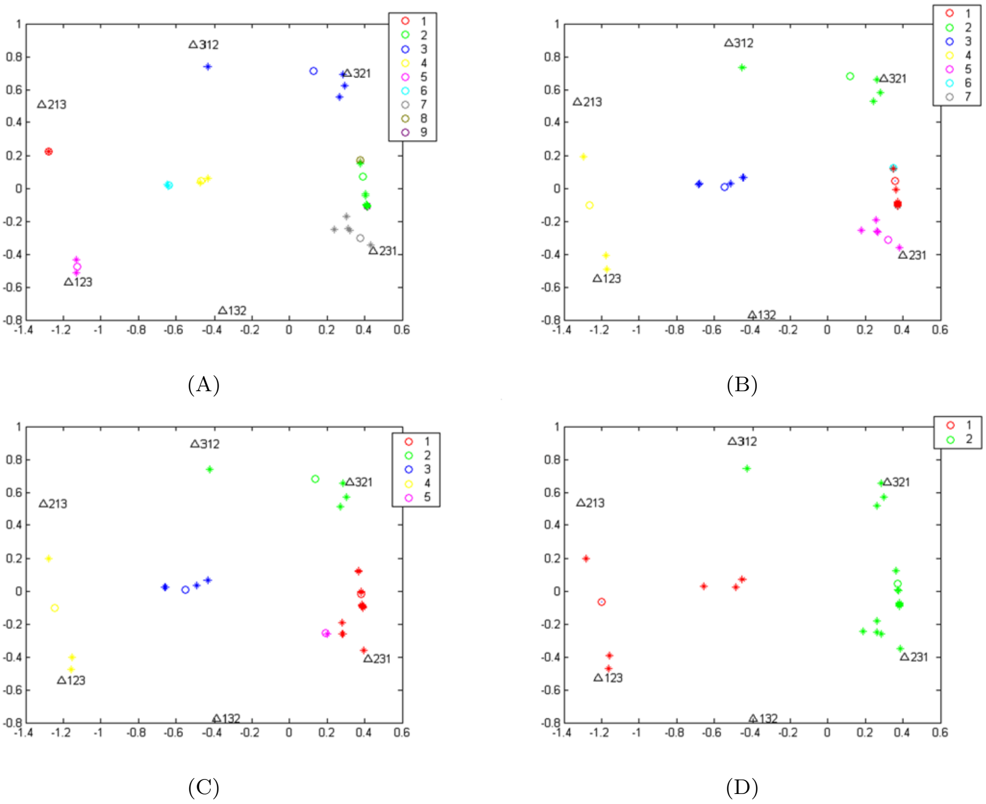



**Figure** 8: *MDS* diagrams for the consistency levels: (A) $e_{max} = 200$; (B) $e_{max} = 500$ (C) $e_{max} = 650$; (D) $e_{max} = 700$.

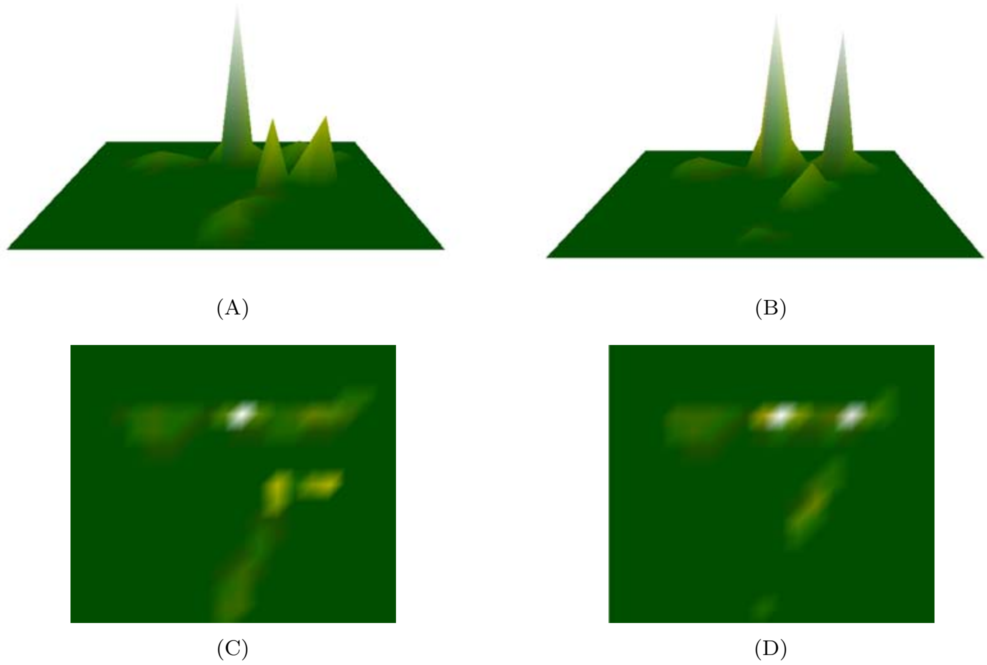


Figure 9: Consensus density diagrams: (A) three–dimensional view for $e_{max} = 200$; (B) three–dimensional view for $e_{max} = 500$; (C) density diagram for $e_{max} = 200$; (D) density diagram for $e_{max} = 500$.