SEARCHING FOR CONSENSUS IN AHP-GROUP DECISION MAKING. A BAYESIAN PERSPECTIVE

ALTUZARRA, Alfredo[1]
MORENO-JIMÉNEZ, José María[1]
SALVADOR, Manuel[1]

[1]Grupo Decisión Multicriterio Zaragoza (http:gdmz.unizar.es)
Facultad de Económicas y Empresariales
Universidad de Zaragoza. Spain

Abstract

This paper sets out to identify the initial positions of the different decision makers who intervene in a group decision making process with a reduced number of actors, and to establish possible consensus paths between these actors. As a methodological support, it employs one of the most widely-known multicriteria decision techniques, namely, the Analytic Hierarchy Process (AHP). Assuming that the judgements elicited by the decision makers follow the so-called multiplicative model (Crawford and Williams, 1985; Altuzarra et al., 1997; Laininen and Hämäläinen, 2003) with log-normal errors and unknown variance, a Bayesian approach is used in the estimation of the relative priorities of the alternatives being compared. These priorities, estimated by way of the median of the posterior distribution and normalised in a distributive manner (priorities add up to one), are a clear example of compositional data that will be used in the search for consensus between the actors involved in the resolution of the problem through the use of Multidimensional Scaling tools.

Keywords: Analytic Hierarchy Process (AHP), Compositional Data, Bayesian Analysis, MDS, Negotiation, Group Decision Making.

* This paper has been partially financed by the multidisciplinary project of the Gobierno de Aragón "*Gobierno electrónico. Toma de decisiones complejas basadas en Internet: e-democracia y e-cognocracia*" (ref. PM2004-052).

1. INTRODUCTION

The philosophical, methodological and technological changes that have arisen in the Knowledge Society in recent years call for the use of new scientific approaches that allow multiactor decision making to be more open and flexible than has traditionally been the case (Moreno-Jiménez et al., 1999, 2001). These approaches must allow the integration of intangible and subjective aspects associated with the human factor in the resolution of the problems.

One of the methodological approaches that answers this kind of necessities is the Analytic Hierarchy Process (AHP) developed by Thomas Saaty in the mid-1970s. Its ability to integrate the small with the large, the individual with the collective, the objective with the subjective and the deterministic with the stochastic so as to capture the interdependencies of the model and its potential to extract the knowledge derived from the scientific resolution of the problem have made it one of the most commonly employed tools in the resolution of complex multi-actor problems.

Assuming a scenario with multiple actors and a common hierarchy, the procedures traditionally used in AHP Group Decision Making are based on the aggregation of judgements (AIJ) or priorities (AIP), which can provoke the rejection of the most dissenting actors. In order to favour the search for a final consensus that adequately represents the individual interests, it is convenient to improve our knowledge of the decisional process (Moreno-Jiménez et al., 2002). It is particularly necessary to have more information about the discrepancies of the actors in order to find out the critical points and the decision opportunities existing in the problem.

In this paper a general framework that facilitates the practical implementation of this process is established. The proposed procedure is based on the identification of the "agreement" and "disagreement" zones among the actors obtained from the analysis of the pairwise comparison matrices that reflect their preferences, by using the fundamental scale proposed by Saaty (1980). To that end, a statistical Bayesian model that describes possible agreements among the actors is used. The proposed framework takes into account the negotiation attitude of the actors, letting them adapt their initial positions in order to favour the establishment of consensus paths to reach a more satisfactory final agreement. Furthermore, the methodology is valid for the analysis of incomplete and/or imprecise pairwise comparison matrices, which gives it more flexibility and realism.

The paper is structured as follows: Section 2 briefly revises the AHP group decision making. In Section 3 the problem is posed and the models that are used are defined. Section 4 analyses the problem of the search for consensus. Finally, Section 5 concludes.

2. AHP GROUP DECISION MAKING

The Analytic Hierarchy Process (AHP) is one of the methodological approaches which allows us to resolve high complexity problems involving multiple scenarios, criteria and actors. It was first proposed in the mid-1970s by Thomas L. Saaty (Saaty, 1977, 1980) and is characterised by the building of a ratio scale corresponding to the priorities of the different alternatives of the problem. To that end, AHP follows four steps: (1) modelisation, (2) valuation, (3) prioritisation and (4) synthesis.

The first of these steps involves the construction of a hierarchy that includes the relevant aspects of the problem (criteria, sub-criteria, attributes and alternatives). The second incorporates the individual preferences (judgements) that reflect the relative importance of the alternatives through a pairwise comparisons matrix, $A_{nxn}=(a_{ij})$, with $a_{ij}\ a_{ji}=1$,

i,j=1,...,n and a_{ij} belonging to the fundamental scale proposed by Saaty (Saaty, 1980). The third step provides the local and global priorities of each element of the hierarchy. The *local priorities*, obtained by some of the prioritisation procedures (the eigenvalue method (Saaty, 1980) and the row geometric mean method (Crawford and Williams, 1985) are the two most commonly employed), measure the priorities of the elements of one level of the hierarchy with respect to its parent node. The *global priorities* of any node of the hierarchy are calculated by applying the Hierarchical Composition Principle (Saaty, 1980) and reflect the priority of any node with respect to the main goal. Finally, in the fourth step, the global priorities of alternatives are synthesised by means of an aggregation procedure (here, the weighted arithmetic and geometric means are the most extensively used), to obtain the *total* or *final priorities* of the alternatives. Using these total priorities, it is possible to rank the alternatives of the problem and to take the most appropriate decisions.
Moreover, in contrast to other multi-criteria decision techniques, AHP allows us to measure the inconsistency of the judgement elicitation process, the *consistency ratio* of Saaty (Saaty, 1980) and the *geometric consistency index* (Crawford and Williams, 1985; Aguarón and Moreno-Jiménez, 2003) being the most used measures.
The flexibility of AHP has allowed its use in group decision making. Moreno et al. (2002) distinguish three possible situations: (i) *Group Decision* where the individuals act jointly by looking for a common decision; (ii) *Negotiated Decision* where each actor solves the problem individually and then the agreement and disagreement zones are analyzed in order to reach a consensus and (iii) *Systemic Decision* where each individual acts independently and a tolerance principle is used to look for a way of integrating all the positions**.** In this paper the Negotiated Decision situation is considered in which it is important to take the negotiation attitude into account in order to reach a satisfactory final consensus.
There are two ways to analyse a group decision problem in the classical literature on AHP (Ramanatham and Ganesh, 1994; Forman and Peniwati, 1998): (i) *Aggregation of Individual Judgements* where a new pairwise comparison matrix for the group is constructed aggregating the individual judgements by means of consensus, voting or statistical procedures such as, for instance, the weighted geometric mean. From this matrix, the priority vector is then calculated following any of the existing prioritisation procedures. (ii) *Aggregation of Individual Priorities* where the individual priorities are aggregated in order to obtain the priority of the group, with the usual aggregation procedure being the weighted geometric mean.
It could occur that some of the decision makers do not agree with the proposed solution. In these cases it is necessary to identify the more divergent opinions in order to establish consensus paths that facilitate a more representative and democratic decision process. So it is necessary to identify the actors and judgements responsible for the lack of consensus.
In this paper a general framework based on a Bayesian statistical procedure is proposed that lets us capture some possible agreements between the decision makers. Taking as a starting point the individual positions of the actors, the procedure adjusts them in order to establish, semi-automatically, some possible consensus paths. This problem is considered as an optimisation problem with restrictions which searches for the minimisation of some discrepancy measures that depend on the kind of problem (Roy, 1985). Furthermore, the procedure makes it possible to analyze incomplete and/or imprecise pairwise comparison matrices which makes the analysis more flexible and realistic.

3. FORMULATION OF THE PROBLEM

Assuming a local (single criterion) context, let $\mathbf{D} = \{D_1,\ldots,D_r\}$ $r \geq 2$ be a group of r decision makers expressing *r* reciprocal judgement matrices $\{R_{nxn}^{(k)};\ k = 1,\ldots, r\}$, corresponding to pairwise comparisons for a set of n alternatives $\{A_1,\ldots,A_n\}$ with regard to the criterion considered, where $R_{nxn}^{(k)} = \left(r_{ij}^{(k)}\right)$ is a positive squared matrix which validates $r_{ii}^{(k)} = 1$, $r_{ji}^{(k)} = \frac{1}{r_{ij}^{(k)}} > 0$ for $i \neq j$. The judgements $r_{ij}^{(k)}$ represent the relative importance to the decision maker D_k of A_i compared to A_j, according to the fundamental scale proposed by Saaty (1980).

Let $\mathbf{v} = \{v_1, \ldots, v_n\}$ ($v_1>0, \ldots, v_n>0$) be the group's (unnormalised) priorities for the alternatives obtained by means of any of the usual prioritisation methods, and let $w_1,\ldots, w_n$ s be their normalised distribution values $\left(\sum_{i=1}^{n} w_i = 1\right)$.

3.1 Statistical model

To obtain the group's priorities from a Bayesian point of view, a multiplicative model with normal logarithmic errors is used, such as is traditionally employed in the stochastic AHP (Ramsay, 1977; de Jong, 1984; Crawford and Williams, 1985; Genest and Rivest, 1994; Alho and Kangas, 1997; Altuzarra et al., 1997; Laininen and Hämäläinen, 2003, …) and which is given by the expression:

$$r_{ij}^{(k)} = \frac{v_i}{v_j} e_{ij}^{(k)}, \quad (i,j) \in J_k;\ k = 1,\ldots,r \tag{3.1}$$

where $J_k \subseteq J = \{(i,j): 1 \leq i < j \leq n\}$ is the comparison set provided by decision maker D_k in the upper triangle of the $R_{nxn}^{(k)}$ matrix and the errors $e_{ij}^{(k)} \sim LN\left(0, \lambda_{ij}^{(k)} \sigma^2\right)$ follow log-normal independent distributions, where $\lambda_{ij}^{(k)} \geq 0$ are provided by the agents and reflect their negotiation attitude.

Taking the logarithms, a regression model with normal errors is obtained given by:

$$y_{ij}^{(k)} = \mu_i - \mu_j + \varepsilon_{ij}^{(k)} \quad (i,j) \in J_k;\ k = 1,\ldots, r \tag{3.2}$$

where $y_{ij}^{(k)} = \log\left(r_{ij}^{(k)}\right)$, $\mu_i = \log(v_i)$ y $\varepsilon_{ij}^{(k)} \sim N\left(0, \lambda_{ij}^{(k)} \sigma^2\right)$. In order to avoid problems of identifications, the alternative A_n ($\mu_n = 0$) is established as the benchmark.

$\lambda_{ij}^{(k)}$ factors are decision maker specifics and determine his flexibility level to adapt to the group's priorities (**v**). So, the bigger (smaller) its value, the bigger (smaller) will be its tolerance and the bigger (smaller) will be the error values $\varepsilon_{ij}^{(k)}$ assumed by the decision maker. These factors, that express the position of each decision maker, are based, habitually, on subjective aspects like his knowledge of the problem (previous experiences) or whether he is interested in reaching an agreement. Initially it is supposed that $\lambda_{ij}^{(k)} = 1$, $\forall i,j,k$ so that the variability is given by the assumed inconsistency level of the group (σ^2).

3.2 Prior distribution

The prior distribution for the vector of log-priorities is given by:

$$\mu = (\mu_1,\ldots,\mu_{n-1}) \mid \sigma^2 \sim N_{n-1}\left(\mathbf{0}_{n-1}, \frac{\sigma^2}{\lambda_\mu} \mathbf{I}_{n-1}\right) \text{ with } \lambda_\mu \geq 0 \tag{3.3}$$

$$\phi = \frac{1}{\sigma^2} \sim G\left(\frac{p}{2}, \frac{a}{2}\right) \tag{3.4}$$

where G(p,a) denotes gamma distribution with mean p/a and variance p/a^2, $\mathbf{0}_{n-1}$ denotes the n-1 dimensional null vector and $\mathbf{I}_{n-1}$ the identity matrix (n-1)x(n-1).
The constants p, a and λ_μ control the influence of these prior distributions on inferences about parameters of the model (3.2), so that the smaller the parameters a and λ_μ are, the less informative are the distributions (3.3) and (3.4). Specifically, the prior distribution of reference arises where $p \to 0$, $a \to 0$ y $\lambda_\mu \to 0$.

3.3 Posterior distribution

In the following discussion, [X] will denote the density of the random variable X and [Y|X] the density of the conditioned distribution Y|X. Let $\mathbf{y}^{(k)} = \left(y_{ij}^{(k)}; (i,j) \in J_k\right)'$ be the vector expressed by the decision maker D_k; k =1,…,r where $J_k \subseteq J = \{(i,j): 1 \le i < j \le n\}$ is the judgement set expressed by D_k and let $|J_k|$ be its cardinal. If all the judgements have been expressed, $|J_k| = \frac{n(n-1)}{2}$. Let $\mathbf{X}^{(k)} = \left(x_{ij}^{(k)}\right)$ be a matrix J_kx(n-1) such that, if the i^{th} component of the vector $\mathbf{y}^{(k)}$ corresponding to the comparison between A_j and A_ℓ with $1 \le j < \ell < n$ then $x_{ij}^{(k)} = 1$, $x_{i\ell}^{(k)} = -1$ and $x_{is}^{(k)} = 0$ for $s \neq j, \ell$ and if $\ell = n$ then $x_{ij}^{(k)} = 1$ and $x_{is}^{(k)} = 0$ for $s \neq j$.
In matrix notation, model (3.2) could be expressed as

$$\mathbf{y}^{(k)} = \mathbf{X}^{(k)}\boldsymbol{\mu} + \boldsymbol{\varepsilon}^{(k)} \text{ ; } k = 1,\ldots, r \tag{3.5}$$

with $\boldsymbol{\varepsilon}^{(k)} = \left(\varepsilon_{ij}^{(k)}\right)' \sim N_{|J_k|}\left(\mathbf{0}_{|J_k|}, \sigma^2\mathbf{D}^{(k)}\right)$ where $\mathbf{D}^{(k)}_{(|J_k| x |J_k|)} = \text{diag}\left(\boldsymbol{\lambda}^{(k)}\right)$ with $\boldsymbol{\lambda}^{(k)} = \left(\lambda_{ij}^{(k)}\right)'$. Let $\mathbf{y} = \left(\mathbf{y}^{(1)'}, \ldots, \mathbf{y}^{(r)'}\right)'$ be an (Nx1) vector that contains logarithms of the judgements expressed by the decision makers, where $N = \sum_{k=1}^{r} |J_k|$ is the total number of the judgements expressed by all the decision makers.
Let $\{\beta^{(1)},\ldots,\beta^{(r)}\}$ be a set of r positive values assigned by the analyst to weight the importance that the opinion of each decision maker has in the aggregate judgement process. These weights have been determined by factors such as social importance, the weight of the group they represent, their position in the firm, etc. Specifically, if all the decision makers have the same importance, then we will take $\{\beta^{(k)} = 1; k = 1,\ldots, r\}$.
Using (3.3)-(3.5), weights $\{\beta^{(1)},\ldots,\beta^{(r)})$ and supposing that the prior distribution is the reference distribution, that is, if $p \to 0$, $a \to 0$ y $\lambda_\mu \to 0$ then, it follows that:

$$\phi|\mathbf{y} \sim G\left(\frac{(N-n+1)}{2}, \frac{\sum_{k=1}^{r} \beta^{(k)}\left(\mathbf{y}^{(k)} - \mathbf{X}^{(k)}\hat{\mathbf{m}}_\mu\right)'\left(\mathbf{D}^{(k)}\right)^{-1}\left(\mathbf{y}^{(k)} - \mathbf{X}^{(k)}\hat{\mathbf{m}}_\mu\right)}{2}\right) \tag{3.6}$$

$$\boldsymbol{\mu}|\mathbf{y} \sim T_{N-n+1}\left(\hat{\mathbf{m}}_\mu, \hat{s}^2\left(\sum_{k=1}^{r} \beta^{(k)}\mathbf{X}^{(k)}{}'\left(\mathbf{D}^{(k)}\right)^{-1}\mathbf{X}^{(k)}\right)^{-1}\right) \tag{3.7}$$

where: $$\hat{\mathbf{m}}_\mu = \left(\sum_{k=1}^{r} \beta^{(k)} \mathbf{X}^{(k)'} \left(\mathbf{D}^{(k)}\right)^{-1} \mathbf{X}^{(k)}\right)^{-1} \left(\sum_{k=1}^{r} \beta^{(k)} \mathbf{X}^{(k)'} \left(\mathbf{D}^{(k)}\right)^{-1} \mathbf{y}^{(k)}\right) \quad (3.8)$$

$$\hat{s}^2 = \frac{\sum_{k=1}^{r} \beta^{(k)} \left(\mathbf{y}^{(k)} - \mathbf{X}^{(k)} \hat{\mathbf{m}}_\mu\right)' \left(\mathbf{D}^{(k)}\right)^{-1} \left(\mathbf{y}^{(k)} - \mathbf{X}^{(k)} \hat{\mathbf{m}}_\mu\right)}{N - n + 1} \quad (3.9)$$

$T_n(\mathbf{m},\mathbf{S})$ denotes a multivariate Student-t with n degrees of freedom, and parameters **m** and **S.** As the degrees of freedom of the model determine the precision of the estimated parameters, we will require that weights $\beta^{(k)}$ satisfy $\sum_{k=1}^{r} \beta^{(k)} |J_k| = N$ so that the degrees of freedom of the model do not change because of factors that are external to the data. Particularly, if every decision maker expresses all the judgements ($\mathbf{X}^{(k)} = \mathbf{X}$, $\mathbf{D}^{(k)} = \mathbf{D}$ $\forall$ k=1,…,r), it follows that:

$$E[\mu|\mathbf{y}] = \hat{\mathbf{m}}_\mu = \frac{\sum_{k=1}^{r} \beta^{(k)} \hat{\mathbf{m}}_\mu^{(k)}}{\sum_{k=1}^{r} \beta^{(k)}} \quad (3.10)$$

where $\left\{ \hat{\mathbf{m}}_\mu^{(k)} = \left(\mathbf{X}^{(k)'} \left(\mathbf{D}^{(k)}\right)^{-1} \mathbf{X}^{(k)}\right)^{-1} \left(\mathbf{X}^{(k)'} \left(\mathbf{D}^{(k)}\right)^{-1} \mathbf{y}^{(k)}\right); k = 1,...,r \right\}$ are the estimators MLE of μ, obtained on the basis of the judgements expressed separately by all the decision makers. It is a weighted mean of the estimators $\hat{\mathbf{m}}_\mu^{(k)}$ with the coefficients $\beta^{(k)}$, which leads to the more important decision makers having more influence on the aggregate judgement process.

$$E[\sigma^2|\mathbf{y}] = \left[r - \frac{2(n+1)}{n(n-1)}\right] \sum_{k=1}^{r} \beta^{(k)} \mathrm{MSE}_k\left(\hat{\mathbf{m}}_\mu\right) \quad (3.11)$$

with $\mathrm{MSE}_k\left(\hat{\mathbf{m}}_\mu\right) = \frac{\left(\mathbf{y}^{(k)} - \mathbf{X}\hat{\mathbf{m}}_\mu\right)' (\mathbf{D})^{-1} \left(\mathbf{y}^{(k)} - \mathbf{X}\hat{\mathbf{m}}_\mu\right)}{n(n-1)/2}$ is a consistency measure when $E[\mu|\mathbf{y}]$ is taken as the priority vector of D_k. So, the most important and the least flexible decision maker will have more influence on the estimation of the group consistency.

3.4 Example

To illustrate the proposed methodology we include the following example about a group decision problem taken from Wang and Xu (1990). Suppose six decision makers that give the following comparison matrices with regard to the same decision problem.

$$R_1 = \begin{bmatrix} 1 & 3 & 5 & 4 & 7 \\ 1/3 & 1 & 3 & 2 & 5 \\ 1/5 & 1/3 & 1 & 1/2 & 3 \\ 1/4 & 1/2 & 2 & 1 & 3 \\ 1/7 & 1/5 & 1/3 & 1/3 & 1 \end{bmatrix} \quad R_2 = \begin{bmatrix} 1 & 4 & 3 & 5 & 8 \\ 1/4 & 1 & 4 & 3 & 6 \\ 1/3 & 1/4 & 1 & 1 & 5 \\ 1/5 & 1/3 & 1 & 1 & 7 \\ 1/8 & 1/6 & 1/5 & 1/7 & 1 \end{bmatrix} \quad R_3 = \begin{bmatrix} 1 & 1/2 & 3 & 2 & 5 \\ 2 & 1 & 5 & 1 & 2 \\ 1/3 & 1/5 & 1 & 2 & 1/2 \\ 1/2 & 1 & 1/2 & 1 & 5 \\ 1/5 & 1/2 & 2 & 1/5 & 1 \end{bmatrix}$$

$$R_4 = \begin{bmatrix} 1 & 3 & 5 & 2 & 6 \\ 1/3 & 1 & 1 & 3 & 2 \\ 1/5 & 1 & 1 & 4 & 5 \\ 1/2 & 1/3 & 1/4 & 1 & 1/2 \\ 1/6 & 1/2 & 1/5 & 2 & 1 \end{bmatrix} \quad R_5 = \begin{bmatrix} 1 & 2 & 6 & 3 & 3 \\ 1/2 & 1 & 2 & 5 & 4 \\ 1/6 & 1/2 & 1 & 1/2 & 1 \\ 1/3 & 1/5 & 2 & 1 & 5 \\ 1/3 & 1/4 & 1 & 1/5 & 1 \end{bmatrix} \quad R_6 = \begin{bmatrix} 1 & 2 & 5 & 4 & 9 \\ 1/2 & 1 & 3 & 2 & 6 \\ 1/5 & 1/3 & 1 & 1 & 2 \\ 1/4 & 1/2 & 1 & 1 & 3 \\ 1/9 & 1/6 & 1/2 & 1/3 & 1 \end{bmatrix}$$

Let r=6, $\mathbf{X}^{(k)} = \mathbf{X}$ be and $J_k = 10\ \forall k$. We will take, furthermore, $\beta^{(k)} = 1$ assigning the same importance to all the decision makers. Tables 3.1 to 3.3 show the results obtained when we analyse, from a Bayesian standpoint, model (3.5) separately for every decision maker, with $\lambda_\mu \to 0$, p=0.1 and a = 0.0177 (Altuzarra et al., 2005). The prior distribution is diffuse and centred on the inconsistency limit proposed by Genest and Rivest (1994).We can also see the consensus priorities, the level of inconsistency of the group (Table 3.1) and the posterior consensus distribution of the most preferred alternatives (Table 3.2) and the preference structures (Table 3.3) taking $\lambda_{ij}^{(k)} = 1\ \forall i,j,k$.

Table 3.1. Posterior medians of the priorities and individuals and consensus inconsistency levels (Credibility interval of 95% in brackets)

	$w_1^{(k)}$	$w_2^{(k)}$	$w_3^{(k)}$	$w_4^{(k)}$	$w_5^{(k)}$	$\sigma^{(k)}$
D_1	0.491 (0.461,0.520)	0.232 (0.211,0.254)	0.092 (0.083,0.103)	0.138 (0.125,0.153)	0.046 (0.041,0.052)	0.259 (0.175,0.439)
D_2	0.480 (0.426,0.535)	0.249 (0.208,0.295)	0.117 (0.096,0.142)	0.120 (0.026,0.032)	0.032 (0.026,0.039)	0.481 (0.325,0.817)
D_3	0.301 (0.237,0.371)	0.318 (0.251,0.391)	0.102 (0.077,0.134)	0.183 (0.140,0.235)	0.092 (0.069,0.121)	0.699 (0.472,1.186)
D_4	0.451 (0.388,0.511)	0.183 (0.147,0.225)	0.210 (0.170,0.257)	0.073 (0.058,0.092)	0.081 (0.064,0.101)	0.565 (0.382,0.959)
D_5	0.407 (0.349,0.463)	0.290 (0.241,0.341)	0.084 (0.068,0.105)	0.147 (0.119,0.180)	0.071 (0.056,0.087)	0.526 (0.355,0.892)
D_6	0.475 (0.462,0.488)	0.261 (0.251,0.271)	0.098 (0.093,0.102)	0.120 (0.115,0.126)	0.046 (0.044,0.048)	0.112 (0.076,0.190)
Consen.	0.437 (0.376,0.504)	0.258 (0.209,0.312)	0.114 (0.089,0.144)	0.128 (0.101,0.160)	0.058 (0.046,0.074)	0.623 (0.552,0.712)

Table 3.2. Posterior probability distributions (%) of the most likely alternatives for each decision maker and consensus distribution + in a P.α problem

	A_1	A_2	A_3	A_4	A_5
D_1	99.90	0.10	0.00	0.00	0.00
D_2	97.10	2.90	0.00	0.00	0.00
D_3	42.60	52.90	0.20	4.30	0.00
D_4	96.60	0.90	2.40	0.10	0.00
D_5	81.50	18.10	0.00	0.40	0.00
D_6	100.00	0.00	0.00	0.00	0.00
Π_α, consensus	100.00	0.00	0.00	0.00	0.00

+ The probability distributions have been obtained from 10000 simulations of the distribution (4.8) for each decision maker and the consensus distribution (3.12)

Table 3.3. Posterior probability distributions (%)
of the preference structure for each decision maker and consensus distribution [+] in a P.γ problem

	12345*	12435	12354	12453	21435	21453	13245	13254
D_1	0.01	99.99	0.00	0.00	0.00	0.00	0.00	0.00
D_2	43.00	57.00	0.00	0.00	0.00	0.00	0.00	0.00
D_3	0.11	27.34	0.03	10.08	44.25	18.18	0.00	0.00
D_4	4.57	0.01	12.57	0.00	0.00	0.00	21.63	61.21
D_5	0.07	89.27	0.00	9.34	1.09	0.23	0.00	0.00
D_6	0.00	100.00	0.00	0.00	0.00	0.00	0.00	0.00
Π_γ, consensus	24.00	75.90	0.00	0.00	0.10	0.00	0.00	0.00

[+] The probability distributions have been obtained from 10000 simulations of the distribution (4.8) for each decision maker and the consensus distribution (3.12)

*Preference structure $A_1 \succ A_2 \succ A_3 \succ A_4 \succ A_5$ where $\succ$ means "preferred over".

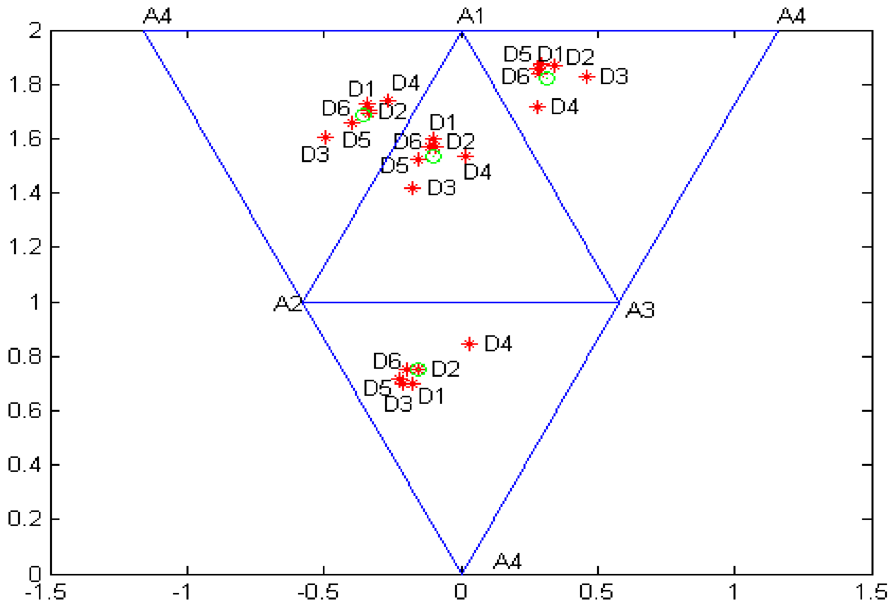

Figure 3.1: Ternary diagrams of the priorities vector suppressing A_5 (o represents consensus priority)

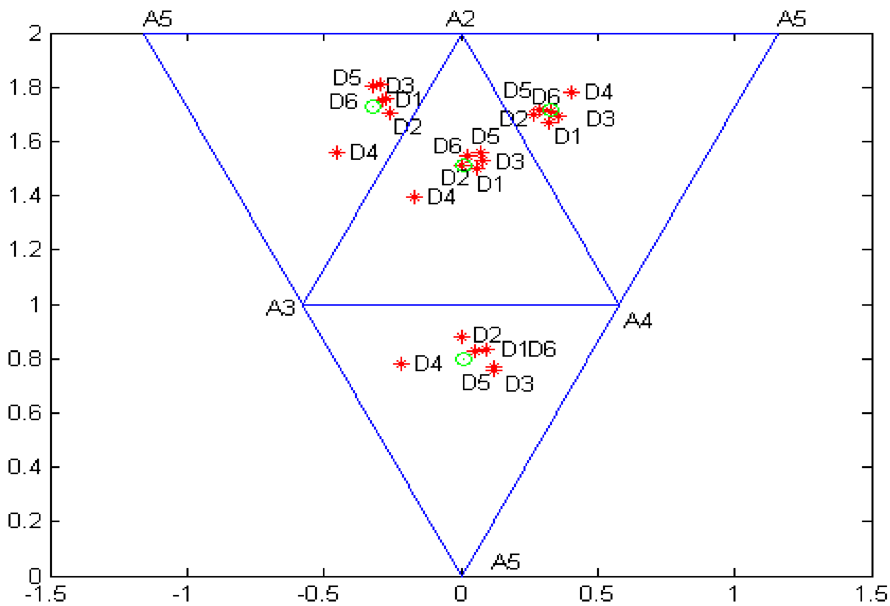

Figure 3.2: Ternary diagrams of the priorities vector suppressing A_1 (o represents consensus priority

Decision makers D_1 and D_6 are the most consistent of the group because their variances are lower (Table 3.1). Also, except the most inconsistent decision maker (D_3), who prefers A_2, the rest of them much prefer A_1 to the others (Table 3.2).
With respect to the rankings, D_1 and D_6 show a very clear preference for the ranking $A_1 \succ A_2 \succ A_4 \succ A_3 \succ A_5$ where $\succ$ means "preferred over" (from now on [12435]). This structure is also chosen by D_5, and, though less strongly, by D_2,. The decision makers who are most clearly in disagreement with the above mentioned preference structure are D_3 and D_4, (Figures 3.1 and 3.2) who are also the most inconsistent of the six (Table 3.1). D_3 opts for the ranking [21435], but not clearly in view of the high probability of rank reversal between alternatives A_1 and A_2, and A_3 and A_5. Finally, D_4 opts for the ranking [13254], but with a high probability of rank reversal between alternatives A_2 and A_3, and A_5 and A_4, as reflected in the not insignificant probabilities of the preference structures [13245] and [12354] (Table 3.3). Furthermore, the partial ranking [12***] is selected by four decision makers (D_1, D_2, D_5, D_6) with almost 100% probability.
From the consensus distribution (Tables 3.2 and 3.3) the following situations stand out as the most preferred: (i) alternative A_1 (99.99%); (ii) the partial ranking [12***] with 100% and (iii) the preference structure [12435] with 75.90%. These conclusions are valid for most of the decision makers.

4. A BAYESIAN APPROACH FOR CONSENSUS BUILDING

After obtaining the posterior consensus distribution of the priorities vector, $\mathbf{v} = \exp(\mu)$, its representativeness is evaluated. Our proposal is to use discrepancy measures, $D(\mathbf{a}^{(k)},\mathbf{v})$, between the above priorities and the judgements expressed by each decision maker ($\mathbf{a}^{(k)}$). These judgements will depend on the kind of decisional problem considered (Roy, 1985); their calculation will allow us to identify the agents who most disagree with the proposal consensus and why. On the basis of this information, we can begin searching for consensus paths that lead, possibly, to a common position. In this section several ways are described to construct the discrepancy measures and a procedure is introduced to obtain, semi-automatically, consensus paths that provide negotiation directions that can lead to wide agreement among decision makers.

4.1 Discrepancy measurements

4.1.1 Predictive discrepancy measurement

This measure seeks to evaluate the level of incompatibility of the vector of judgements expressed by each decision maker with the priorities vector $\mathbf{v}$. For this we use the Bayesian p-value

$$D_{pred}(\mathbf{a}^{(k)},\mathbf{v}) = P\left[f^{(k)}_{consenso}(\mathbf{a}) \leq f^{(k)}_{consenso}(\mathbf{a}^{(k)})\right] \quad (4.1)$$

where $f^{(k)}_{consenso}(\mathbf{a}) = \int f^{(k)}(\mathbf{a}|\mathbf{v},\sigma^2,\lambda^{(k)})\pi_{consenso}(\mathbf{v},\sigma^2)d\mathbf{v}d\sigma^2$ is the predictive distribution of the vector of judgements $\mathbf{a}$, built on the basis of the consensus distribution $\pi_{consenso}$ given for expressions (3.6)-(3.9).
Standard calculations show that

$$D_{pred}(\mathbf{a}^{(k)},\mathbf{v}) = P\left[F_{|J_k|,p+N} \leq \frac{1}{|J_k|}\frac{\left(\mathbf{y}^{(k)} - \mathbf{X}^{(k)}\mathbf{m}_\mu\right)'\left(\mathbf{X}^{(k)'}\mathbf{S}_\mu\mathbf{X}^{(k)} + \mathbf{D}^{(k)}\right)^{-1}\left(\mathbf{y}^{(k)} - \mathbf{X}^{(k)}\mathbf{m}_\mu\right)}{s^2}\right] \quad (4.2)$$

where $F_{n,m}$ denotes the Snedecor's F distribution with n and m degrees of freedom.

Discrepancy (4.1) measures the complementary of the minimum credibility level such that, if the discrepancy is very small (credibility level very high) it is a sign that the judgement is in the smallest density zone of the predictive distribution $f^{(k)}_{consenso}(\mathbf{a})$ and, therefore, it is more improbable that the judgement has been expressed by a decision maker whose preferences are described by the consensus distribution $\pi_{consenso}$.
Likewise, with a discrepancy measurement similar to (4.1), the judgements expressed by each decision maker that are most incompatible with the consensus proposed by the group would be identified. It would enough to calculate, for each judgement $a^{(k)}_{ij}$, the Bayesian p-value:

$$P\left[f^{(k)}_{consenso}\left(a_{ij}\right) \le f^{(k)}_{consenso}\left(a^{(k)}_{ij}\right)\right] \tag{4.3}$$

where $f^{(k)}_{consenso}\left(a_{ij}\right) = \int f^{(k)}\left(a_{ij} \mid \mathbf{v}, \sigma^2, \lambda^{(k)}\right) \pi_{consenso}\left(\mathbf{v}, \sigma^2\right) d\mathbf{v} d\sigma^2$ is the predictive density of the a_{ij} judgement, built from the above Bayesian statistic model.
Standard calculations show that (4.3) is given by:

$$D_{pred}(a^{(k)}_{ij}, \mathbf{v}) = P\left[F_{1,p+N} \ge \frac{\left(y^{(k)}_{ij} - m_{\mu,i} + m_{\mu,j}\right)^2}{s^2\left(s_{\mu,ii} + s_{\mu,jj} - 2s_{\mu,ij}\right)}\right] = P\left[\left|t_{p+N}\right| \ge \frac{\left|y^{(k)}_{ij} - m_{\mu,i} + m_{\mu,j}\right|}{s\sqrt{\left(s_{\mu,ii} + s_{\mu,jj} - 2s_{\mu,ij}\right)}}\right] \tag{4.4}$$

where $\mathbf{S}_\mu = \left(\sum_{k=1}^{r} \beta^{(k)} \mathbf{X}^{(k)'} \left(\mathbf{D}^{(k)}\right)^{-1} \mathbf{X}^{(k)}\right)^{-1}$.

4.1.2. Discrepancy measurement in P.α problems

Type α decisional problems (P.α) are understood (Roy, 1985) to be those in which we try to determine the "best" alternative. In this case, the discrepancy measure tries to evaluate the agreement level of each decision maker with respect to the consensus opinion about the alternative most preferred by the group. The consensus opinion is given by:

$$\pi_{\alpha,consenso,i} = P\left[v_i = \max_{1\le j\le n} v_j\right] = \int_{\{\mu_i = \max_{1\le j\le n} \mu_j\}} \pi_{consenso}(\boldsymbol{\mu}) d\boldsymbol{\mu}\text{; } i=1,\ldots,n \tag{4.5}$$

where $\pi_{consenso}(\mu)$ is the consensus distribution density of (3.7).
As these probabilities cannot be calculated analytically, Monte Carlo methods are used to evaluate them. The discrepancy measurement is given by:

$$D_\alpha(\mathbf{a}^{(k)}, \mathbf{v}) = \sqrt{\left(\boldsymbol{\pi}^{(k)'}_\alpha \mathbf{S}_\alpha \boldsymbol{\pi}^{(k)}_\alpha\right) + \left(\boldsymbol{\pi}'_{consenso} \mathbf{S}_\alpha \boldsymbol{\pi}_{consenso}\right) - 2\left(\boldsymbol{\pi}^{(k)'}_\alpha \mathbf{S}_\alpha \boldsymbol{\pi}_{consenso}\right)} \tag{4.6}$$

where $\mathbf{S}_\alpha = (s_\alpha(A_i, A_j))$ is a similarity measurement between A_i and A_j and $\boldsymbol{\pi}^{(k)}_\alpha = \left(\pi^{(k)}_{\alpha,1}, \ldots, \pi^{(k)}_{\alpha,n}\right)'$ with: $\pi^{(k)}_{\alpha,i} = \int_{\left\{\mu^{(k)}_i = \max_{1\le j\le n} \mu^{(k)}_j\right\}} \pi^{(k)}\left(\boldsymbol{\mu}^{(k)}\right) d\boldsymbol{\mu}^{(k)}$; i=1,…,n (4.7)

$\pi^{(k)}(\mu^{(k)})$ being a distribution $T_{p+|J_k|}\left(\mathbf{m}^{(k)}_\mu, s^{(k)2}\mathbf{S}^{(k)}_\mu\right)$: (4.8)

$$\mathbf{m}^{(k)}_\mu = \left(\mathbf{X}^{(k)'}\left(\mathbf{D}^{(k)}\right)^{-1}\mathbf{X}^{(k)}\right)^{-1}\left(\mathbf{X}^{(k)'}\left(\mathbf{D}^{(k)}\right)^{-1}\mathbf{y}^{(k)}\right) \qquad \mathbf{S}^{(k)}_\mu = \left(\mathbf{X}^{(k)'}\left(\mathbf{D}^{(k)}\right)^{-1}\mathbf{X}^{(k)}\right)^{-1}$$

$$s^{(k)2} = \frac{a + \left(\mathbf{y}^{(k)} - \mathbf{X}^{(k)}\mathbf{m}^{(k)}_\mu\right)'\left(\mathbf{D}^{(k)}\right)^{-1}\left(\mathbf{y}^{(k)} - \mathbf{X}^{(k)}\mathbf{m}^{(k)}_\mu\right)}{p + |J_k|}$$

which is the *posterior* distribution of the log-priorities from D_k, obtained from a model analogous to (3.2). One way of building the similarity s_α is given by Kronecker's delta $s_\alpha(A_i,A_j) = 1$ if i=j and 0 otherwise, with $\mathbf{S}_\alpha = \mathbf{I}_n$.

4.1.3 Example (continuation)

Carrying on with the above example (3.4), Tables 4.1.a and 4.1.b show the discrepancy measurement values (4.1) and (4.6) for each decision maker (Table 4.1.a) and also the discrepancy measurement (4.4) for each expressed judgement (Table 4.1.b). In addition, Figures 4.2 and 4.4 show perceptual diagrams obtained after applying a classical[1] MDS to the distance matrices that are given by: $d_{ij} = \sqrt{\left(\boldsymbol{\pi}_t^{(i)'}\mathbf{S}_t\boldsymbol{\pi}_t^{(i)}\right)+\left(\boldsymbol{\pi}_t^{(j)'}\mathbf{S}_t\boldsymbol{\pi}_t^{(j)}\right)-2\left(\boldsymbol{\pi}_t^{(i)'}\mathbf{S}_t\boldsymbol{\pi}_t^{(j)}\right)}$ $i,j\in\{1,...,r\}$, with t=α in the case of Figure 4.4 and t=γ in the case of Figure 4.2.
The probabilities vector $\boldsymbol{\pi}_\alpha^{(i)}$ is that used to calculate the discrepancy measurement (4.6) and $\mathbf{S}_\gamma=(s_\gamma(R,R'))_{R,R'\in PS}$, where PS is the set of preference structures, is a similarity measure between any of the preference structures and $\boldsymbol{\pi}_\gamma^{(k)} = \left(\pi_{\gamma,R}^{(k)}\right)_{R\in S}$, where $\pi_{\gamma,R}^{(k)} = \int_{\left\{\mu_{i_1}^{(k)}\geq\mu_{i_2}^{(k)}\geq...\geq\mu_{i_r}^{(k)}\right\}}\pi^{(k)}\left(\boldsymbol{\mu}^{(k)}\right)d\boldsymbol{\mu}^{(k)}$ with $\pi^{(k)}(\boldsymbol{\mu}^{(k)})$ being the distribution used in (4.7).

It can be seen, as was to be expected, that the most discordant decision makers of the group are D_3 and D_4 (Tables 4.1a and 4.1b for judgements). In the case of D_3 there are disagreements in all the measures used and these are due to his higher level preference for alternative A_2 (Table 3.2 and Figure 4.4). In the case of decision maker D_4, the disagreements are found in measure (4.1) and this is due to his higher level preference for structure [13254] (Table 3.3 and Figure 4.2).

Table 4.1.a. Global discrepancies of each decision maker with consensus distribution

Discrep. \ Decis. maker	D_1	D_2	D_3	D_4	D_5	D_6
$D_{pred}(\mathbf{a}^{(k)},\mathbf{v})$	0.0154	0.4779	0.8999	0.9161	0.4137	0.0003
$D_\alpha(\mathbf{a}^{(k)},\mathbf{v})$	0.0014	0.0453	0.7693	0.0469	0.2390	0.0000
$D_\gamma(\mathbf{a}^{(k)},\mathbf{v})$	0.1267	0.1712	0.5000	0.7486	0.2411	0.1267
$D_{consistency}(\mathbf{a}^{(k)},\mathbf{v})$	0.1398	0.4046	0.7192	0.7788	0.3651	0.0712

Table 4.1.b. Discrepancies of the judgements expressed by each decision maker with consensus distribution

Discrepancy/Decision maker	D_1	D_2	D_3	D_4	D_5	D_6
$D_{pred}(a_{12}^{(k)},\mathbf{v})$	0.6204	0.8126	0.9387	0.6204	0.1992	0.1992
$D_{pred}(a_{13}^{(k)},\mathbf{v})$	0.3172	0.2986	0.2986	0.3172	0.5101	0.3172
$D_{pred}(a_{14}^{(k)},\mathbf{v})$	0.1969	0.448	0.59	0.59	0.1562	0.1969
$D_{pred}(a_{15}^{(k)},\mathbf{v})$	0.0816	0.0834	0.4671	0.2672	0.8401	0.2262
$D_{pred}(a_{23}^{(k)},\mathbf{v})$	0.3395	0.6228	0.7792	0.7912	0.1504	0.3395
$D_{pred}(a_{24}^{(k)},\mathbf{v})$	0.0024	0.4681	0.7169	0.4681	0.8406	0.0024
$D_{pred}(a_{25}^{(k)},\mathbf{v})$	0.1575	0.3696	0.7759	0.7759	0.1173	0.3696
$D_{pred}(a_{34}^{(k)},\mathbf{v})$	0.6244	0.148	0.7901	0.9778	0.6244	0.148
$D_{pred}(a_{35}^{(k)},\mathbf{v})$	0.4971	0.8532	0.9617	0.8532	0.6965	0.0335
$D_{pred}(a_{45}^{(k)},\mathbf{v})$	0.3717	0.9242	0.7954	0.9754	0.7954	0.3717

[1] The program used was cmdscale from MATLAB 6.5

4.2 Consensus paths

Having studied the decision makers' standpoints and their disagreements with the group's standpoint, we will go on to search for a consensus between them so that the decision making process can be as representative and democratic as possible. To do so, we are going to follow two different procedures where, one way or another, the direct participation of the agents in the negotiation process is required: (i) the consensus of the agents on the basis of the information provided about the most discordant judgements and (ii) the negotiation between the agents on the basis of the information supplied by the different consensus paths which have been obtained, semi-automatically, for different scenarios. All this will be carried out using an inverse sensibility analysis of the problem that will culminate in a process of discussion and debate with the aim of fixing new standpoints, in principle closer to each other. In the following sub-sections both procedures are studied in depth.

4.2.1 Revision of judgements

One way to reach a common standpoint is to identify and revise the judgements that most disagree with the proposed consensus. We must compare the expressed values of these judgements with the values predicted by the consensus distribution. Predictions are made using marginals from the predictive distribution of the log-judgements $\{\mathbf{y}^{(k)}\}$, built from the *posterior* distribution (3.7)

$$\mathbf{y}^{(k)} \sim T_{p+N}\left(\mathbf{X}^{(k)}\mathbf{m}_{\mu}, s^2\left(\mathbf{X}^{(k)}\mathbf{S}_{\mu}\mathbf{X}^{(k)\prime}+\mathbf{I}_{J_k}\right)\right) \tag{4.9}$$

where $\mathbf{m}_{\mu}$, $\mathbf{S}_{\mu}$ and s^2 are distribution parameters (4.8). From this, Bayesian credibility intervals can be constructed to provide a consensus path between the decision makers.

4.2.1.1 Example (continuation)

Using the discrepancy measurement (4.4) it can be seen (Table 4.1.b) that the judgements that are most in disagreement with the proposed consensus are $a_{35}^{(3)}$, $a_{34}^{(4)}$ and $a_{45}^{(4)}$ and, somewhat less, $a_{12}^{(3)}$. Table 4.2 shows the 95% Bayesian credibility intervals and the *posterior* median of the predictive distribution of consensus, both obtained from distribution (4.9).

Table 4.2. Bayesian credibility intervals of 95% and posterior median of the disagreement judgements from consensus by the predicted consensus distribution

Judgement	Cuantil 2.5	Median	Cuantil 97.5	Observed value
$a_{12}^{(3)}$	0.4607	1.6998	6.2722	0.5
$a_{34}^{(4)}$	0.2403	0.8867	3.2717	4
$a_{35}^{(3)}$	0.5275	1.9465	7.1826	0.5
$a_{45}^{(4)}$	0.5950	2.1953	8.1007	0.5

It can be observed, particularly, that the judgements expressed by decision makers D_3 and D_4 are outside of the above intervals, with the exception of $a_{12}^{(3)}$ which is located near the lower limit. Comparing these values, D_3 and D_4 could analyse whether or not to change their values on these judgements towards the directions indicated by the intervals in order to reach a greater consensus.

4.2.2. Semi-automatic generation of consensus paths

One alternative way of establishing consensus paths consists of incorporating the negotiation attitude of the agents into the model and to simulate different scenarios, that is to say, to establish negotiation processes which would allow each decision maker to smooth or harden his original standpoint up to a certain limit, analyzing how the consensus distribution (3.12) would change.
For this, we would propose optimisation problems in the following way:

$$\text{Min } \|D(\mathbf{a}^{(k)},\mathbf{v}(\lambda^{(1)},\dots,\lambda^{(r)}))\|p,\beta \quad (4.10)$$

$$\text{subject to } \lambda_{min} = 1 - \frac{p_{max}}{100} \leq \lambda^{(k)} \leq 1 + \frac{p_{max}}{100} = \lambda_{max} \text{ with } 0<p_{max}<100$$

where $\mathbf{v}(\lambda^{(1)},\dots,\lambda^{(r)})$ denotes the vector of consensus priorities obtained from model (3.1) with $\lambda_{ij}^{(k)} = \lambda^{(k)}$ $(1\leq i<j\leq n, k=1,\dots,r)$, D being one of the discrepancy measurements defined above, p the norm selected and $\beta=\{\beta^{(1)},\dots,\beta^{(r)}\}$ the set of weights assigned by the analyst. We use the Tchevichev norm (L_∞) and $\beta^{(k)}=1$ $\forall k$. Parameter p_{max} determines the maximum percentage of consistency of the group (σ^2) and the flexibility of attitude of each decision maker. Thus the bigger p_{max}, the greater the decision makers' negotiation capacity and, therefore, the greater the possibility of reaching agreements between the decision makers.

4.2.2.1 Example (continuation)

Figures 4.1 and 4.2 show the evolution of the *posterior* consensus distributions of the preference structures estimated for different scenarios, according to the values of the negotiation attitude (limits p_{max}). Figure 4.1 shows these distributions as a line diagram. Figure 4.2 shows the same, but in a perceptual diagram calculated from the predictive disagreements between decision makers as was described in Subsection 4.1. Below we describe the results obtained with the two discrepancies measures:
(i) D_{pred}: It can be seen that, the bigger the negotiation limit, the bigger the *posterior* probability of consensus on the preference structure [12435] (Figures 4.1 and 4.2) and the smaller the predictive disagreements of D_3 and D_4 since they adapt to the situation by increasing their tolerance levels $\lambda^{(k)}$. Moreover, the estimated value of priority of alternative A_1, the group's favourite, also increases.

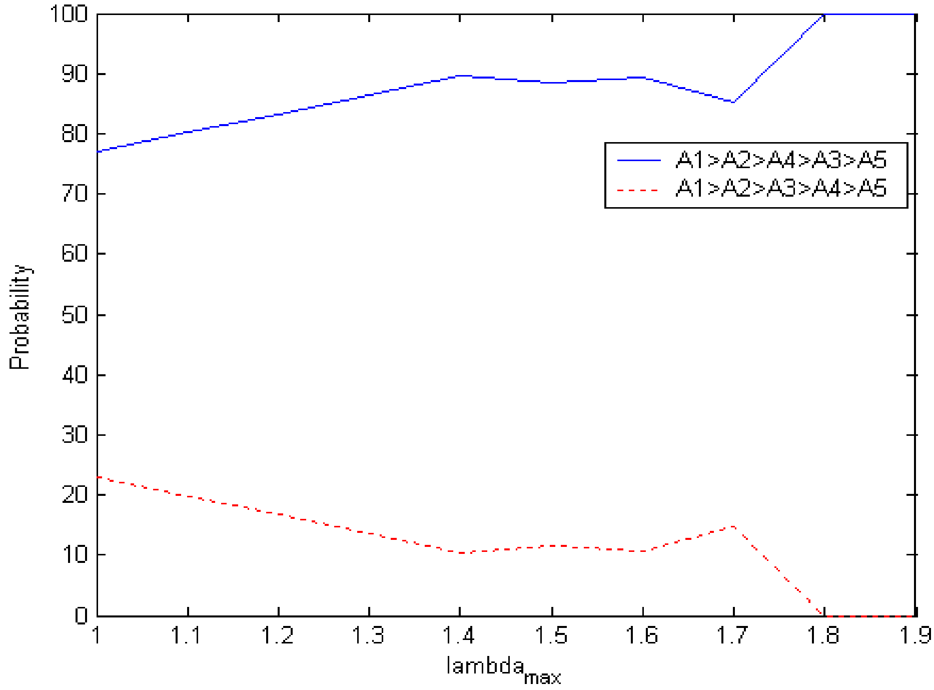

Figure 4.1: Sensibility analysis of the posterior consensus distribution of the preference structures when is wanted a predictive consensus

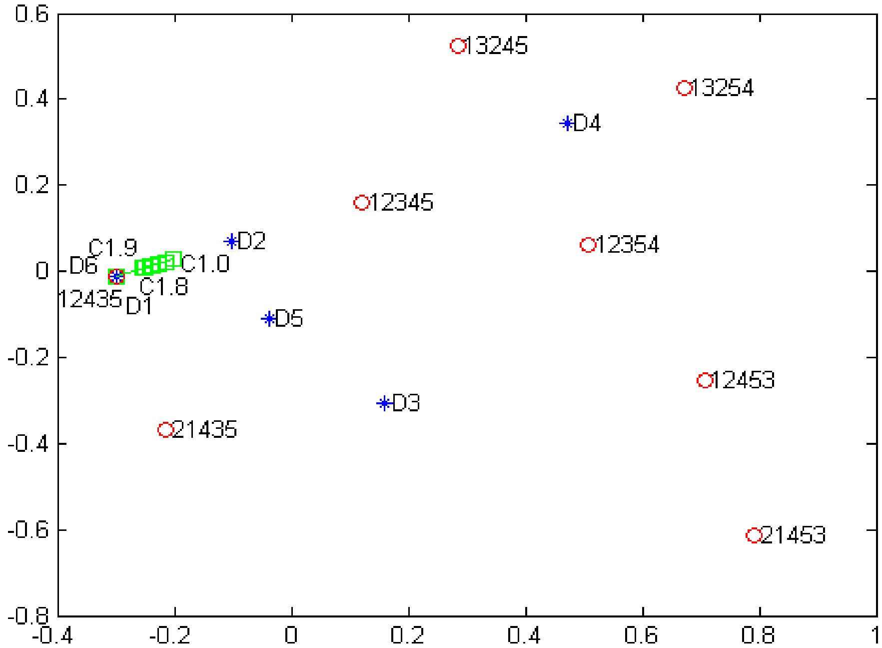

Figure 4.2: Perceptual diagram of the evolution of the posterior distributions of the preference structures from each decision maker and the consensus distributions when is wanted a predictive consensus (o12345 is the preference structure [12345]; □C1.2 is consensus distribution when λ_{max}=1.2 and *D1 decision maker D_1)

(ii) D_α: If the problem is analysed from the point of view of discrepancy D_α, the results shown in Figures 4.3 and 4.4 are obtained. Their significance is similar to that described for discrepancy, D_{pred}. In this case, it can be observed that the consensus distribution about the most preferred alternative (A_1) is quite stable. Only when the negotiation limits are wide, does the consensus probability of alternative A_2, the favourite of decision maker D_3, increase. This priority tends to increase when the negotiation limits widen. Consensus is reached, in this case, by increasing the tolerance levels of all the decision makers except D_3.

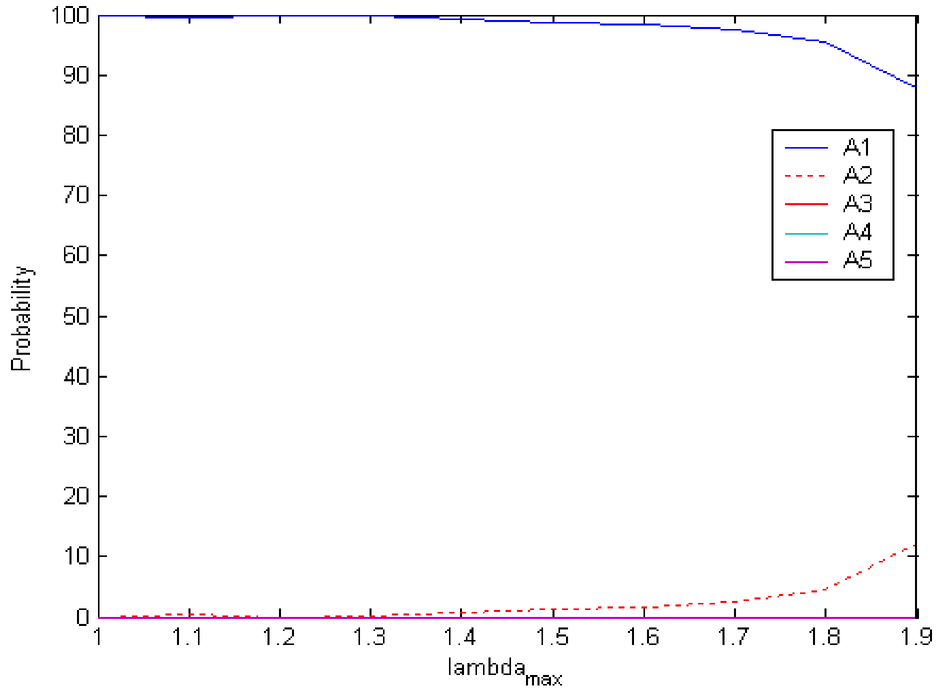

Figure 4.3: Sensibility analysis of the posterior consensus distribution of the most likely alternative when is wanted an alfa consensus

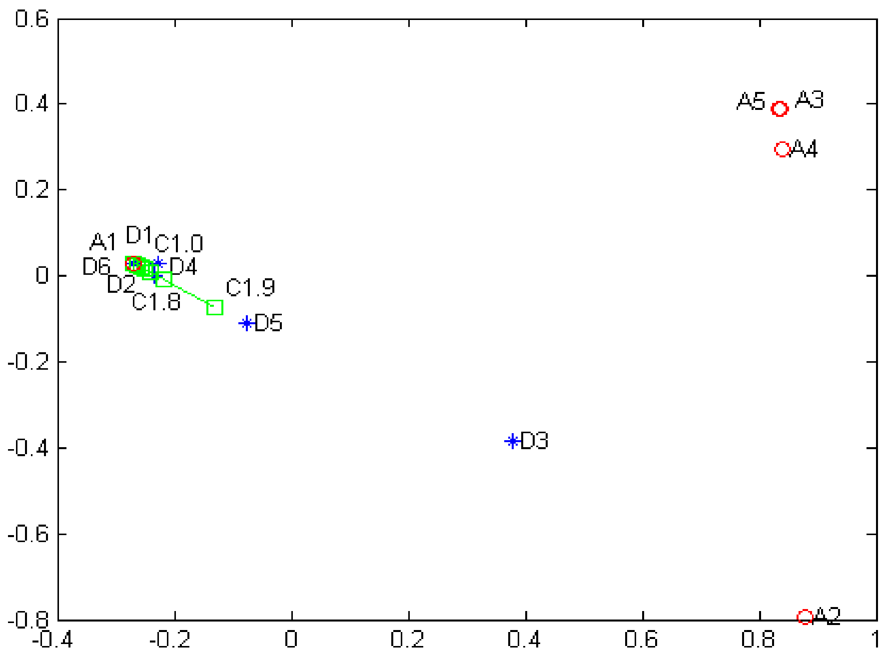

Figure 4.4: Perceptual diagram of the evolution of the posterior distributions of the most likely alternative from each decision maker and the consensus distributions when is wanted an alfa consensus (oA1 represent A1 alternative; □C1.7 is consensus distribution when λ_{max}=1.7 and *D1 decision maker D_1)

The sensibility study carried out suggests various strategies for reaching a consensus. The first strategy would consist of the discordant decision makers D_3 and D_4 taking more flexible standpoints and/or reviewing their discrepant judgements. In this case, the consensus initially described would be maintained and would be compatible with the standpoints of all the decision makers. With respect to the P.α problem, it should be noted that there is a big consensus that A_1 is selected in first place. Decision maker D_3 is an exception because he supports alternative A_2 instead of A_1, but in spite of this A_1 has, at least, for this decision maker, a probability of 42.6% of being selected in the first place. The determination of the appropriate procedure in each case is an open question that would require a deeper debate within the group.

5. Conclusions

In this paper a Bayesian methodology for the semiautomatic search for consensus building in AHP group decision making has been introduced. The proposed procedure consists of two steps. In the first step, the existing individual discrepancies are analysed by using a Bayesian approach based on the multiplicative log-normal errors traditionally used in the stochastic AHP. Using the information obtained, some procedures are proposed to search for consensus between the actors involved in the decision making process. Some of them are based on the modifications of the more divergent individual judgements; others determine semiautomatic consensus paths by using the negotiation attitude of the decision makers.

The methodology allows us to analyse to what extent it is possible to aggregate the information provided by each decision maker without loosing the consistency of judgements. Furthermore, and in the case of a lack of agreement between the decision makers, the methodology allows us to find the reasons for this lack, by analysing the disagreements within the groups. This information is used in the construction of different scenarios from which negotiation and judgements revision processes can be initiated in order to reach consensus.

The methodology is appropriate if the number of decision makers is not large (10 or less) because, in these cases, the negotiation processes would be more fluid. If the number is large, the search for a consensus process would be based on the identification of the existing homogeneous opinion groups, and the negotiation process would be carried out between representative agents of each group. In order to identify these homogeneous groups it would be necessary to build cluster algorithms adapted to this situation. This is our current research line (Altuzarra et al., 2005) and its results will be reported elsewhere.

References

AGUARÓN, J. and MORENO-JIMÉNEZ, J.M. (2003): The Geometric Consistency Index. Approximated Thresholds. *European Journal of Operational Research*; 147 (1), p. 137-145.

ALHO, J.M. and KANGAS, J. (1997) Analyzing Uncertainties in Experts' Opinions of Forest Plan Performance. *Forest Science*; 43, p. 521–528.

ALTUZARRA, A.; MORENO-JIMÉNEZ, J. M.; SALVADOR, M. (1997): Una Aproximación Bayesiana en la Incorporación de Juicios en el Proceso Analítico Jerárquico (AHP). *Actas de la XI Reunión Asepelt-España*, Bilbao.

ALTUZARRA, A.; MORENO, J.M. and SALVADOR, M. (2005). A Bayesian Priorization Procedure for AHP-Group Decision Making (submitted).

CRAWFORD, G. and WILLIAMS, C. (1985): A Note on the Analysis of Subjective Judgment Matrices. *Journal of Mathematical Psychology,* 29, p. 387–405.

DE JONG, P. (1984): A statistical approach to Saaty´s scaling method for priorities. *Journal of Mathematical Psychology*, 28, p. 467-478.

FORMAN, E. and PENIWATI, K. (1998): Aggregating individual judgments and priorities with the Analytic Hierarchy Process. *European Journal of Operational Research*, p.108, 165–169.
GENEST, C. and RIVEST, L.P. (1994): A Statistical Look at Saaty's Method of Estimating Pairwise Preferences Expressed on a Ratio Scale. *Journal of Mathematical Psycholog* , 38, p. 477-496.
LAININEN, P. and HÄMÄLÄINEN, R.P.(2003): Analyzing AHP-matrices by regression. *European Journal of Operation Research,* 2003, 148, p.514-524.
MORENO-JIMÉNEZ, J.M. (1997): Priorización y toma de decisiones ambientales. *Actas del 1er. Encuentro Iberoamericano de Evaluación y Decisión Multicriterio*, 113-145. Santiago de Chile (Chile). Conferencia plenaria.
MORENO-JIMÉNEZ, J.M.; AGUARÓN, J ; ESCOBAR, M.T. AND TURÓN, A.(1999): The Multicriteria Procedural Rationality on Sisdema. *European Journal of Operational Research*, 119 (2), p. 388–403.
MORENO-JIMÉNEZ, J.M.; AGUARÓN, J. AND ESCOBAR, M.T. (2001): Metodología científica en valoración y selección ambiental. *Pesquisa Operacional*, 21, p. 3-18 (Revista de la Sociedad Brasileña de IO).
MORENO-JIMÉNEZ, J.M.; AGUARÓN, J.; ESCOBAR, M.T. (2002): Decisional Tools for Consensus Building in AHP-Group Decision Making. *12th. Mini Euro Conference*. Bruselas.
RAMANATHAN, R. and GANESH, L.S.(1994): Group Preference Aggregation Methods employed in AHP: An Evaluation and Intrinsic Process for Deriving Members' Weightages, *European Journal of Operational Research*, 79, p. 249–265.
RAMSAY, J.O. (1977): Maximum likelihood estimation in multidimensional scaling. *Psychometrika* **42**, 241-266.
ROBERT, C.P. and CASELLA, G.(1999): *Monte Carlo Statistical Methods.* New-York, Springer-Verlag.
ROY, B. (1985): *Mèthodologie Multicritére d'aide à la Décision.* Gestión Económica.
SAATY, T.L. (1977): A Scaling Method for Priorities in Hierarchical Structures. *Journal of Mathematical Psychology* 15/3, 234-281.
SAATY, T.L.(1980): *Multicriteria Decision Making: The Analytic Hierarchy Process*. Mc Graw-Hill, New York. 2ª impresión 1990, RSW Pub. Pittsburgh)
TANNER, M.A. and WONG, W.H. (1987): The calculation of posterior distributions by data augmentation (with discussion). *Journal of the American Statistical Association*, 82, p.528-550.
WANG, Y. M. and XU, N. R. (1990): Linear Programming Method for Solving the Group-AHP. *Journal of Southeast University*, 20 (6), 58-63.