

Error and Complexity of Random Walk Monte Carlo Radiosity

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Abstract—In this paper, we study the error and complexity of the discrete random walk Monte Carlo technique for Radiosity, both the shooting and gathering methods. We show that the shooting method exhibits a lower complexity than the gathering one, and under some constraints, it has a linear complexity. This is an improvement over a previous result that pointed to an $O(n \log n)$ complexity. We give and compare three unbiased estimators for each method, and obtain closed forms and bounds for their variances. We also bound the expected value of the Mean Square Error (MSE). Some of the results obtained are also shown to be valid for the nondiscrete gathering case. We also give bounds for the variances and MSE for the infinite path length estimators; these bounds might be useful in the study of the biased estimators resulting of cutting off the infinite path.

Index Terms—Rendering, radiosity, Monte Carlo, random walk.



1 INTRODUCTION

TO study the complexity of a Radiosity algorithm we should first establish a measure of error which gives us an invariant between the different scenes to compare. In Monte Carlo algorithms, the obvious choice for such a measure is the variance of the radiosity estimator for each patch (which manifests itself as noise in the final image), and also the area weighted combination of those variances, which, for an unbiased estimator, is simply the expected value of the Mean Square Error (MSE). Then, to study the complexity, we should study the variation in cost when we change the different parameters but keep the same variances or MSE. A difficulty that arises is which exact solution we should use to measure the error. The obvious choice would be the radiosity function over the whole scene, solution of the integral equation (Rendering equation). However, this exact solution is never known, and as in radiosity, we have to discretize the environment so as to obtain the equations system, a possibility is to consider the error made respective to the exact solution of the Radiosity equations system. And it results that, although those exact solutions are almost never known (except for very simple environments), the expected value of the error made respective to those solutions can be computed (or at least bounded). We should also account somehow for the error of discretizing, although this error decreases to zero when the discretization becomes finer and finer. Another issue that appears when studying random walk Monte Carlo radiosity is how to define the trajectory of the particles carrying the light power in the simulation. If a particle must follow the Form-Factor probabilities all along its trajectory, the impinging point on each patch must be forgotten and a new exit point and a direction must be selected randomly

(this is done implicitly in [16], [7]). The expected values of the simulation are then the exact solutions of the Radiosity system. On the other hand, following pure Particle-Tracing [12], we use the same impinging point as the exit point for a particle. Thus, the trajectory of a particle is described by point to patch differential Form-Factors, not by the patch to patch Form-Factor matrix, and the classical results on random walk in [8], [14] are not directly applicable. Of course, both kind of simulations converge to the same result when the average size of a patch decreases.

In this paper, we will study both shooting random walk and gathering random walk, that is, random walk from the sources and random walk from the patches. When solving a system of equations through random walk [8], [14], we are provided with a dual set of solutions. The *direct* solution corresponds to the gathering approach, and in the nondiscrete case it would correspond to path-tracing (without the shadow rays). The solution proceeds sending paths from the patches of interest to *gather* energy when a source is hit. On the other hand, the adjoint system solves for the *importance* or contribution that each source has for illuminating a given patch [13]. The *adjoint* solution will appear when solving the adjoint system of equations by the direct method. As we are interested in solving for the importance of the sources, the paths are traced from them to gather importance from the patches. This can be interpreted as if particles carrying energy were *shot* from the sources and followed through the environment, to distribute their energy. Orthogonal to the shooting and gathering duality is the duality between the radiosity and the power system, although this orthogonality does not introduce any new independent solution. We can solve either system by shooting or gathering, but it is usual to solve the power system by shooting and the radiosity system by gathering, and we will do so in this paper. Another important point is whether the estimators are biased or not. We will limit ourselves to unbiased estimators, although biased estimators exist and should be investigated.

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It must also be remarked that apart from random walk Monte Carlo radiosity (or rather, *discrete* random walk Monte Carlo radiosity) other Monte Carlo radiosity methods exist, such as the Stochastic Radiosity method [11], global Monte Carlo methods [15], and others. Path-tracing [9], and even distributed ray-tracing [4], [19], [20], can be considered as the limiting case of gathering random walk for the nondiscrete case. Bidirectional ray-tracing [18], [10] is a mixture of nondiscrete shooting and gathering. Shooting random walk methods have also been investigated in the limiting, nondiscrete case [5], [6].

The complexity of shooting random walk Monte Carlo radiosity has been understood to this date to be of order $O(n \log n)$, where n is the number of patches in the scene [17]. However, in [17], the study of the complexity is limited to the particular case where the total area in the scene is kept constant. A second point is that variances are bounded respective to a postulated (by physical reasons) maximum radiance (or radiosity) in the scene, but no bound is given for such radiance. So this result is, to our understanding, incomplete. In this paper, we will try to complete it in the following ways: first, we will define a random walk unbiased estimator for the radiosity of a patch and give a *closed formula* (not just a bound) for the variance of this estimator. Second, we will give a bound for the maximum radiosity on the scene and a bound for all the variances. Third, we will show that Monte Carlo random walk radiosity can be considered (under some restrictions) to be $O(n)$. And last, we will define other unbiased estimators and compare their efficiencies. We also give bounds for an infinite path length estimator; this might be useful to study the efficiency of the biased estimators obtained when the path is cut off. All this will be done in the next section, dedicated to the study of shooting random walk. In Section 3, we will study gathering random walk. Closed forms for the variances of estimators analogous to the ones for the shooting case will be given, with bounds for the MSE, and a study of the complexity will be done, showing that it exhibits a higher complexity than the shooting case. And we will show that a formula for the variance also applies to the nondiscrete case. In Section 4, some results are presented that confirm our theoretical findings. Finally, in Section 5, we present our conclusions and future research.

2 SHOOTING RANDOM WALK

In this section, we study three unbiased and one infinite path length estimators for the incoming power ϕ_i on a patch i . The unbiased estimators were introduced by Shirley in [17], although in a slightly different form. We will prove that the unbiased estimators are indeed unbiased, after which we present a closed-form expression for the variance on each patch and bounds for both the variance and the MSE. The complexity is also obtained.

2.1 An Unbiased Random-Walk Estimator for the Incoming Power: The $\frac{\Phi_T}{1-R_i}$ Estimator

Let us first consider what the expected value of any unbiased Monte Carlo estimator should be for the incoming

power on a patch. Let us suppose that the initial power of source s is Φ_s , ϕ_i is the incoming power on patch i , F_{kl} denotes the Form-Factor from patch k to patch l , and R_k denotes the reflectance of patch k . Then we have, by developing the Power system in Neumann series:

$$\begin{aligned} \phi_i = & \sum_s \Phi_s F_{si} + \sum_s \sum_h \Phi_s F_{sh} R_h F_{hi} \\ & + \sum_s \sum_h \sum_j \Phi_s F_{sh} R_h F_{hj} R_j F_{ji} + \dots \end{aligned}$$

This can be expressed as:

$$\phi_i = \phi_i^{(1)} + \phi_i^{(2)} + \phi_i^{(3)} + \dots$$

where

$$\phi_i^{(1)} = \sum_s \Phi_s F_{si}, \quad \phi_i^{(2)} = \sum_s \sum_h \Phi_s F_{sh} R_h F_{hi},$$

$$\phi_i^{(3)} = \sum_s \sum_h \sum_j \Phi_s F_{sh} R_h F_{hj} R_j F_{ji},$$

and so on. That is, $\phi_i^{(1)}$ represents the power arrived directly from the sources, $\phi_i^{(2)}$ represents the power arrived after one bounce, and so on.

Let us now consider the following simulation. A source s is selected with probability $\frac{\Phi_s}{\Phi_T}$, where Φ_T is the sum of all powers for all the sources. A particle exits from this source according to the Form-Factors probability (simulated selecting a random exit point and a random direction [15]), and goes to patch j with probability F_{sj} . Then it survives or dies according to the probabilities $(R_j, 1 - R_j)$. The expected length of the trajectory (or path) γ is bounded by $\frac{1}{1-R_{max}}$ [17], where R_{max} is the maximum of the reflectivities. Now let us define for patch i and path γ the family of random variables $\hat{\phi}_i^{(1)}, \hat{\phi}_i^{(2)}, \hat{\phi}_i^{(3)}, \dots$ in the following way:

All of those random variables are initially null. If the path γ happens to finish on patch i at length l (that is, the particle dies at length l), then the value of $\hat{\phi}_i^{(l)}$ is set to $\frac{\Phi_T}{1-R_i}$.

Let us also define a new random variable $\hat{\phi}_i$ as:

$$\hat{\phi}_i = \hat{\phi}_i^{(1)} + \hat{\phi}_i^{(2)} + \hat{\phi}_i^{(3)} + \dots$$

PROPOSITION 2.1. *For all $l \geq 1$, the random variable $\hat{\phi}_i^{(l)}$ is an unbiased estimator for the power arrived to patch i after l bounces, and $\hat{\phi}_i$ is an unbiased estimator of the total incoming power arrived to patch i after any number of bounces.*

PROOF. Applying the definition of expected value, and remembering that the probability of selecting source s is $\frac{\Phi_s}{\Phi_T}$, the probability of landing on patch i just after leaving source s is F_{si} and the probability of dying on patch i is $1 - R_i$, we have

$$E\left(\hat{\phi}_i^{(1)}\right) = \sum_s \frac{\Phi_T}{(1-R_i)} \frac{\Phi_s}{\Phi_T} F_{si}(1-R_i) = \phi_i^{(1)}$$

Now, to go from a source s to i in a two length path we can pass through any patch h , and survive in it with probability R_h , so we have

$$E\left(\hat{\phi}_i^{(2)}\right) = \sum_s \sum_h \frac{\Phi_T}{(1-R_i)} \frac{\Phi_s}{\Phi_T} F_{sh} R_h F_{hi}(1-R_i) = \phi_i^{(2)}$$

and so on. Then, we have

$$\begin{aligned} E\left(\hat{\phi}_i\right) &= E\left(\hat{\phi}_i^{(1)} + \hat{\phi}_i^{(2)} + \dots\right) = E\left(\hat{\phi}_i^{(1)}\right) + E\left(\hat{\phi}_i^{(2)}\right) + \dots \\ &= \phi_i^{(1)} + \phi_i^{(2)} + \dots = \phi_i \end{aligned}$$

□

We have considered above the simulation for only one path or particle. For N particles we would use N such estimators and average by N . Or alternatively, we could consider the sum of N estimators with $\frac{1}{N}$ of the total energy each. In any case, and due to the way we select the sources, upon their power, each particle carries the same quantity of energy.

PROPOSITION 2.2. *The variance of the estimator $\hat{\phi}_i$ is:*

$$\text{Var}\left(\hat{\phi}_i\right) = \phi_i \left(\frac{\Phi_T}{(1-R_i)} - \phi_i \right)$$

PROOF. Considering the definition for variance and the fact that our estimator is unbiased:

$$\begin{aligned} \text{Var}\left(\hat{\phi}_i\right) &= \text{Var}\left(\hat{\phi}_i^{(1)} + \hat{\phi}_i^{(2)} + \dots\right) \\ &= E\left(\left(\hat{\phi}_i^{(1)} + \hat{\phi}_i^{(2)} + \dots\right)^2\right) - \left(E\left(\hat{\phi}_i\right)\right)^2 \\ &= E\left(\hat{\phi}_i^{(1)2}\right) + E\left(\hat{\phi}_i^{(2)2}\right) + \dots \\ &\quad + 2 \sum_{1 \leq n < m} E\left(\hat{\phi}_i^{(n)} \hat{\phi}_i^{(m)}\right) - \phi_i^2 \end{aligned} \quad (1)$$

But each term of the form $E\left(\hat{\phi}_i^{(n)} \hat{\phi}_i^{(m)}\right)$ is null, because if a particle dies at length n on patch i it cannot die again on the same patch at length m . So we have

$$\text{Var}\left(\hat{\phi}_i\right) = E\left(\hat{\phi}_i^{(1)2}\right) + E\left(\hat{\phi}_i^{(2)2}\right) + \dots - \phi_i^2$$

But

$$\begin{aligned} E\left(\hat{\phi}_i^{(1)2}\right) &= \sum_s \left(\frac{\Phi_T}{(1-R_i)} \right)^2 \frac{\Phi_s}{\Phi_T} F_{si}(1-R_i) \\ &= \frac{\Phi_T}{(1-R_i)} \phi_i^{(1)} \end{aligned}$$

$$\begin{aligned} E\left(\hat{\phi}_i^{(2)2}\right) &= \sum_s \sum_h \left(\frac{\Phi_T}{(1-R_i)} \right)^2 \frac{\Phi_s}{\Phi_T} F_{sh} R_h F_{hi}(1-R_i) \\ &= \frac{\Phi_T}{(1-R_i)} \phi_i^{(2)} \end{aligned}$$

and so on. Then we obtain

$$\begin{aligned} \text{Var}\left(\hat{\phi}_i\right) &= \frac{\Phi_T}{(1-R_i)} \left(\phi_i^{(1)} + \phi_i^{(2)} + \dots \right) - \phi_i^2 \\ &= \frac{\Phi_T}{(1-R_i)} \phi_i - \phi_i^2 = \phi_i \left(\frac{\Phi_T}{(1-R_i)} - \phi_i \right) \end{aligned}$$

□

For the radiosity, our estimator is simply $\hat{B}_i = E_i + \frac{R_i}{A_i} \hat{\phi}_i$, where A_i is the area of patch i and E_i the initial emittance. Then, if $b_i = B_i - E_i$ is the reflected radiosity, we have

COROLLARY 2.3. *The variance of the estimator \hat{B}_i is:*

$$\text{Var}\left(\hat{B}_i\right) = b_i \left(\frac{\Phi_T R_i}{A_i(1-R_i)} - b_i \right)$$

PROOF.

$$\begin{aligned} \text{Var}\left(\hat{B}_i\right) &= \frac{R_i^2}{A_i^2} \text{Var}\left(\hat{\phi}_i\right) = \frac{R_i^2}{A_i^2} \phi_i \left(\frac{\Phi_T}{(1-R_i)} - \phi_i \right) \\ &= b_i \left(\frac{\Phi_T R_i}{A_i(1-R_i)} - b_i \right) \end{aligned} \quad (2)$$

□

2.2 A Global Bound for All the Variances

PROPOSITION 2.4. *For all patches i*

$$\text{Var}\left(\hat{B}_i\right) \leq \mathcal{B} \frac{\Phi_T R_{\max}^2}{A_{\min}(1-R_{\max})}$$

where $\mathcal{B} = \max_s \left(\frac{E_s}{(1-R_s)} \right)$, s indexes the sources, A_{\min} is the minimum patch area and R_{\max} the maximum reflectivity.

PROOF. From $b_i \leq R_{\max} \mathcal{B}$, (see Appendix), and from the value of the variance we have for all i :

$$\text{Var}\left(\hat{B}_i\right) \leq b_i \frac{\Phi_T R_i}{A_i(1-R_i)} \leq \mathcal{B} \frac{\Phi_T R_{\max}^2}{A_{\min}(1-R_{\max})}$$

□

The above results are for a single particle. For N independent particles, the variances must be divided by N , and the bound results in (keeping the same name for the N particles estimator):

COROLLARY 2.5.

$$\text{Var}(\hat{B}_i) \leq \mathcal{B} \frac{\Phi_T R_{max}^2}{NA_{min}(1-R_{max})}$$

This means that we can always set a number of paths N so that the variance of any patch is below any pre-established threshold.

2.3 Complexity

Consider the cost of intersecting a particle with a scene composed of n_s surfaces and n_p patches. The expected number of segments of a path is bounded by $\frac{1}{1-R_{max}}$ (maximum expected length). The cost of intersecting a line with the n_s surfaces in the scene is bounded by $\log n_s$ [2] or n_s , depending on the existence or not of a hierarchical structure of the scene (we suppose the cost of intersecting any surface is bounded). This will provide us with the nearest intersection. Now, the cost of picking the right patch within the nearest intersected surface depends on how patches are organized. We can consider two cases: A bounded cost, as with a regular grid, and a hierarchical structure of patches within the surface, where the cost is logarithmic, that is $O(\log \frac{n_p}{n_s})$, because the patches are distributed over all the surfaces. So the complexity of intersecting plus picking a patch is the maximum of both complexities. Then we have the following cases for the cost C_1 for one path:

- 1) structured scene, bounded cost for picking a patch within a surface

$$C_1 = O(\log n_s (1 - R_{max})^{-1})$$

- 2) structured scene, hierarchical structure of patches within a surface

$$C_1 = O\left(\max\left(\log n_s, \log \frac{n_p}{n_s}\right) (1 - R_{max})^{-1}\right)$$

- 3) Nonstructured scene, bounded cost for picking a patch within a surface

$$C_1 = O(n_s (1 - R_{max})^{-1})$$

- 4) Nonstructured scene, hierarchical structure of patches within a surface

$$C_1 = O\left(\max\left(n_s, \log \frac{n_p}{n_s}\right) (1 - R_{max})^{-1}\right)$$

Now, from the previous section, given a bound V for all variances we can find the number of paths N to fulfill this bound:

$$N \geq V^{-1} A_{min}^{-1} (1 - R_{max})^{-1} R_{max}^2 \Phi_T \mathcal{B}$$

Therefore, the total cost C_T of the N paths, which is C_1 times N , is given by:

- 1) structured scene, bounded cost for picking a patch within a surface

$$C_T = O\left(V^{-1} A_{min}^{-1} (1 - R_{max})^{-2} R_{max}^2 \Phi_T \mathcal{B} \log n_s\right)$$

- 2) structured scene, hierarchical structure of patches within a surface

$$C_T = O\left(V^{-1} A_{min}^{-1} (1 - R_{max})^{-2} R_{max}^2 \Phi_T \mathcal{B} \max\left(\log n_s, \log \frac{n_p}{n_s}\right)\right)$$

- 3) nonstructured scene, bounded cost for picking a patch within a surface

$$C_T = O\left(V^{-1} A_{min}^{-1} (1 - R_{max})^{-2} R_{max}^2 \Phi_T \mathcal{B} n_s\right)$$

- 4) nonstructured scene, hierarchical structure of patches within a surface

$$C_T = O\left(V^{-1} A_{min}^{-1} (1 - R_{max})^{-2} R_{max}^2 \Phi_T \mathcal{B} \max\left(n_s, \log \frac{n_p}{n_s}\right)\right)$$

Now imagine the following scenarios:

- 1) Increasing k times the number of surfaces retaining the same minimum area. In this case we can also suppose the number of patches gets increased by k , so the quotient $\frac{n_p}{n_s}$ remains constant. That is, $O(A_{min}) = O(1)$, $O(\frac{n_p}{n_s}) = O(1)$.
- 2) Dividing each existing patch into k equal patches. In this case we do not increase the number of surfaces but A_{min} gets divided by k . That is, $O(A_{min}) = O(n_p^{-1})$, $O(\frac{n_p}{n_s}) = O(n_p)$.

Then, in scenario 1, cases 1 and 2 are $O(\log n_s)$, and cases 3 and 4 are $O(n_s)$. As the quotient $\frac{n_p}{n_s}$ remains constant, this means $O(\log n_p)$ or $O(n_p)$.

In scenario 2, cases 1 and 3 are $O(n_p)$, and 2 and 4 are $O(n_p \log n_p)$.

To sum up, when we do not modify the minimum area and the ratio $\frac{n_p}{n_s}$, we obtain complexity $O(\log n_p)$ or $O(n_p)$ depending on the structuring of the scene. When we add patches via evenly dividing existing patches into new ones, if the cost of picking a patch from within a surface is bounded the complexity is $O(n_p)$, if not it is $O(n_p \log n_p)$. This is represented in Table 1.

As a general conclusion we think that, if we restrict ourselves to the case of bounded cost for picking a patch (cases 1 and 3), it is permitted to speak of linearity. And this does

TABLE 1
COMPLEXITY FOR THE DIFFERENT CASES AND SCENARIOS FOR SHOOTING RANDOM WALK.

Case vs Scenario	1	2
1	$O(\log n_p)$	$O(n_p)$
2	$O(\log n_p)$	$O(n_p \log n_p)$
3	$O(n_p)$	$O(n_p)$
4	$O(n_p)$	$O(n_p \log n_p)$

not depend on the existence of a hierarchical structure of the scene (see case 3). But take into account that: First, almost all the parameters in the scene must be kept constant, for instance, adding an object with a higher reflectivity can break our bound, adding a surface with less area than the minimum also breaks our bound, no power can be added, and so on. Second, the variances are computed with respect to the exact solution of the Radiosity equations system, not respective to the exact solution of the Rendering integral equation. But the majority of Monte Carlo and non-Monte Carlo approaches also take the same reference solution. What we have done here is, given a problem, the integral equation of radiosity, approximate it with a new one, the Radiosity equation system, and obtain the expected error when solving this equation system using random walk. How close our approximated problem is to the original one remains out of the scope of this paper (a theoretical study of the discretization error is found in [1]). We must say, however, that for scenario 2 the discretization error decreases each time we divide the patches (this is not necessarily true for scenario 1). This means linearity for scenario 2 respective to the exact solution of the Rendering integral equation.

2.4 Expected Value of the Mean Square Error

In this section, we will bound the expected value of the MSE. To this objective we must first bound $\sum_i \phi_i$.

PROPOSITION 2.6. *The following bound holds*

$$\sum_i \phi_i \leq \frac{\Phi_T}{1 - R_{\max}}$$

PROOF. We have first

$$\begin{aligned} \sum_i \phi_i &= \sum_i \left(\sum_s \Phi_s F_{si} + \sum_s \sum_h \Phi_s F_{sh} R_h F_{hi} + \dots \right) \\ &= \sum_s \Phi_s \sum_i F_{si} + R_{\max} \sum_s \Phi_s \sum_i \sum_h F_{sh} F_{hi} + \dots \end{aligned}$$

But

$$\sum_h F_{sh} F_{hi} = (F \times F)_{si} = F_{si}^2$$

and as the power of a stochastic matrix is also a stochastic matrix (see for instance [3]), we have $\sum_i F_{si}^2 = 1$. The same happens with any power, so:

$$\begin{aligned} \sum_i \phi_i &\leq \sum_s \Phi_s + R_{\max} \sum_s \Phi_s + R_{\max}^2 \sum_s \Phi_s + \dots \\ &= \Phi_T (1 + R_{\max} + R_{\max}^2 + \dots) = \frac{\Phi_T}{1 - R_{\max}} \end{aligned} \quad (3)$$

□

PROPOSITION 2.7. *For the expected value of the MSE, the bound holds*

$$E(\text{MSE}) \leq \frac{\Phi_T^2}{A_T A_{\min}} \frac{R_{\max}^2}{(1 - R_{\max})^2}$$

PROOF. Applying (2), we obtain:

$$\begin{aligned} E(\text{MSE}) &= \frac{1}{A_T} \sum_i A_i \text{Var}(\hat{B}_i) \\ &= \frac{1}{A_T} \sum_i A_i \frac{R_i^2}{A_i^2} \phi_i \left(\frac{\Phi_T}{(1 - R_i)} - \phi_i \right) \\ &\leq \frac{1}{A_T A_{\min}} \frac{R_{\max}^2}{(1 - R_{\max})} \Phi_T \sum_i \phi_i \\ &\leq \frac{\Phi_T^2}{A_T A_{\min}} \frac{R_{\max}^2}{(1 - R_{\max})^2} \end{aligned}$$

□

COROLLARY 2.8. *For N particles, we have*

$$E(\text{MSE}) \leq \frac{\Phi_T^2}{N A_T A_{\min}} \frac{R_{\max}^2}{(1 - R_{\max})^2} \quad (4)$$

If we take the expected value of the MSE as a measure of the error upon which to study the complexity, from (4), we obtain linearity again, following the same discussion as in Section 2.3. An interesting conclusion from (4) is that adding surfaces keeping the product $A_T A_{\min}$ constant, the expected value of the MSE remains below the same bound.

2.5 Other Unbiased Estimators

2.5.1 The $\frac{\Phi_T}{R_i}$ Estimator

Let us now define for patch i and path γ another family of random variables $\hat{\phi}_i^{(1)}, \hat{\phi}_i^{(2)}, \hat{\phi}_i^{(3)}, \dots$ in the following way:

All of those random variables are initially null. If the particle happens to survive on patch i at length l , then the

value of $\hat{\phi}_i^{(l)}$ is set to $\frac{\Phi_T}{R_i}$. Let us also define a new random variable $\hat{\phi}'_i$ as:

$$\hat{\phi}'_i = \hat{\phi}_i^{(1)} + \hat{\phi}_i^{(2)} + \hat{\phi}_i^{(3)} + \dots$$

PROPOSITION 2.9. *For all $l \geq 1$, the random variable $\hat{\phi}_i^{(l)}$ is an unbiased estimator for the power arrived to patch i after l bounces, and $\hat{\phi}'_i$ is an unbiased estimator for the total incoming power arrived to patch i after any number of bounces.*

The proof follows the one of Proposition 2.1.

PROPOSITION 2.10. *The variance of the estimator $\hat{\phi}'_i$ is*

$$\text{Var}(\hat{\phi}'_i) = \phi_i \left(\Phi_T \left(\frac{1}{R_i} + 2\xi_i \right) - \phi_i \right)$$

where ξ_i is the expected value of the total incoming power on patch i due to emission of a unit power on the same patch i (or also the incident radiosity or irradiance due to a unit emittance).

PROOF. We will use the decomposition of (1). But now

the terms of the form $E(\hat{\phi}_i^{(n)} \hat{\phi}_i^{(m)})$ are no longer null, because if a particle survives at length n on

patch i it can also survive on it at length m . We must then obtain the value of those terms.

$$\begin{aligned} E\left(\hat{\phi}_i^{(n)} \hat{\phi}_i^{(m)}\right) &= \sum_s \sum_{h_1} \cdots \sum_{h_{n-1}} \sum_{h_{n+1}} \cdots \sum_{h_{m-1}} \frac{\Phi_T}{R_i} \frac{\Phi_T}{R_i} \cdot \\ &\quad \frac{\Phi_s}{\Phi_T} F_{sh_1} R_{h_1} \cdots F_{h_{n-1}i} R_i \cdot \\ &\quad F_{ih_{n+1}} R_{h_{n+1}} \cdots F_{h_{m-1}i} R_i \\ &= \phi_i^{(n)} \Phi_T \sum_{h_{n+1}} \cdots \sum_{h_{m-1}} F_{ih_{n+1}} R_{h_{n+1}} \cdots F_{h_{m-1}i} \\ &= \phi_i^{(n)} \Phi_T \xi_i^{(m-n)} \end{aligned}$$

where $\xi_i^{(m-n)}$ is the expected value of the incoming power on patch i after $m - n$ bounces due to a unit power on the same patch i (or also the incident radiosity or irradiance after $m - n$ bounces due to a unit emittance). But we have

$$\sum_{n < m} \xi_i^{(m-n)} = \sum_{1 \leq n} \xi_i^{(n)} = \xi_i$$

The first sum is for all integers greater than n , the second for all integers greater or equal than 1. We obtain

$$\begin{aligned} \sum_{1 \leq n} \left(\sum_{n < m} E\left(\hat{\phi}_i^{(n)} \hat{\phi}_i^{(m)}\right) \right) &= \sum_{1 \leq n} \phi_i^{(n)} \Phi_T \sum_{n < m} \xi_i^{(m-n)} \\ &= \Phi_T \xi_i \sum_{1 \leq n} \phi_i^{(n)} = \Phi_T \xi_i \phi_i \end{aligned}$$

On the other hand we have

$$\begin{aligned} E\left(\hat{\phi}_i^{(1)2}\right) &= \sum_s \left(\frac{\Phi_T}{R_i} \right)^2 \frac{\Phi_s}{\Phi_T} F_{si} R_i = \frac{\Phi_T}{R_i} \phi_i^{(1)} \\ E\left(\hat{\phi}_i^{(2)2}\right) &= \sum_s \sum_h \left(\frac{\Phi_T}{R_i} \right)^2 \frac{\Phi_s}{\Phi_T} F_{sh} R_h F_{hi} R_i = \frac{\Phi_T}{R_i} \phi_i^{(2)} \end{aligned}$$

and so on. Then we finally obtain

$$\begin{aligned} \text{Var}(\hat{\phi}_i') &= \frac{\Phi_T}{R_i} (\phi_i^{(1)} + \phi_i^{(2)} + \cdots) + 2\Phi_T \xi_i \phi_i - \phi_i^2 \\ &= \phi_i \left(\Phi_T \left(\frac{1}{R_i} + 2\xi_i \right) - \phi_i \right) \end{aligned}$$

□

For the radiosity, our estimator is simply $\hat{B}_i = E_i + \frac{R_i}{A_i} \hat{\phi}_i'$, and so

COROLLARY 2.11. *The variance for the estimator \hat{B}_i' is*

$$\text{Var}(\hat{B}_i') = b_i \left(\frac{\Phi_T R_i}{A_i} \left(\frac{1}{R_i} + 2\xi_i \right) - b_i \right)$$

PROOF.

$$\begin{aligned} \text{Var}(\hat{B}_i') &= \frac{R_i^2}{A_i^2} \phi_i \left(\Phi_T \left(\frac{1}{R_i} + 2\xi_i \right) - \phi_i \right) \\ &= b_i \left(\frac{\Phi_T R_i}{A_i} \left(\frac{1}{R_i} + 2\xi_i \right) - b_i \right) \end{aligned}$$

□

The quantity $R_i \xi_i$ is the indirect importance [13] that patch i has to illuminate itself (this means that for a source of importance with value 1 for patch i and zero elsewhere, the total importance for patch i will be $1 + R_i \xi_i$). A bound for ξ_i is given by (see Appendix)

$$\xi_i \leq \frac{1}{1 - R_i} \quad (5)$$

However, we can expect $\xi_i \ll 1$ for the following reason. Given (3), we have that the total incoming power for all patches due to a unit power is bounded by $\frac{1}{1 - R_{\max}}$. The average incoming power per patch is then $\frac{1}{n_p(1 - R_{\max})}$. We can suppose ξ_i be of this order of magnitude. Obviously, the average incoming power decreases to zero when $n_p \rightarrow \infty$. Using now the bound in (5) we have

PROPOSITION 2.12. *The following bound holds*

$$\text{Var}(\hat{B}_i') \leq b_i \left(\frac{\Phi_T(1 + R_i)}{A_i(1 - R_i)} - b_i \right)$$

PROOF. Using the bound for ξ_i in (5) we obtain

$$\begin{aligned} \text{Var}(\hat{B}_i') &\leq b_i \left(\frac{\Phi_T R_i}{A_i} \left(\frac{1}{R_i} + \frac{2}{1 - R_i} \right) - b_i \right) \\ &= b_i \left(\frac{\Phi_T(1 + R_i)}{A_i(1 - R_i)} - b_i \right) \end{aligned} \quad (6)$$

□

This variance can be bounded in the same way as in Section 2.2 and obtain the same linearity results as in Section 2.3. Now, to obtain the expected value of the MSE, we proceed as in Section 2.4 and arrive to:

PROPOSITION 2.13. *The following bound holds*

$$E(\text{MSE}) \leq \frac{\Phi_T^2}{A_T A_{\min}} \frac{R_{\max}(1 + R_{\max})}{(1 - R_{\max})^2}$$

2.5.2 The Φ_T Estimator

Now we will define a third family of random variables for patch i and path $\gamma : \hat{\phi}_i''^{(1)}, \hat{\phi}_i''^{(2)}, \hat{\phi}_i''^{(3)}, \dots$ in the following way:

All of those random variables are initially null. If the particle happens to hit patch i at length l , then the value of

$\hat{\phi}_i''^{(l)}$ is set to Φ_T . Let us also define a new random variable $\hat{\phi}_i''$ as:

$$\hat{\phi}_i'' = \hat{\phi}_i''^{(1)} + \hat{\phi}_i''^{(2)} + \hat{\phi}_i''^{(3)} + \cdots$$

PROPOSITION 2.14. For all $l \geq 1$, the random variable $\hat{\phi}_i^{(l)}$ is an unbiased estimator for the power arrived to patch i after l

bounces, and $\hat{\phi}_i''$ is an unbiased estimator of the total incoming power arrived to patch i after any number of bounces.

The proof follows the one of Proposition 2.1. Using an analogous proof as for Proposition 2.10, we have:

PROPOSITION 2.15. The variance of the estimator $\hat{\phi}_i''$ is given by

$$\text{Var}\left(\hat{\phi}_i''\right) = \phi_i(\Phi_T(1 + 2R_i\xi_i) - \phi_i)$$

For the radiosity, our estimator is simply $\hat{B}_i' = E_i + \frac{R_i}{A_i}\hat{\phi}_i''$, and so

COROLLARY 2.16.

$$\text{Var}(\hat{B}_i') = b_i\left(\frac{R_i\Phi_T}{A_i}(1 + 2R_i\xi_i) - b_i\right) \quad (7)$$

And bounding ξ_i with $\frac{1}{1-R_i}$ we obtain

PROPOSITION 2.17.

$$\text{Var}(\hat{B}_i') \leq b_i\left(\frac{R_i(1 + R_i)\Phi_T}{A_i(1 - R_i)} - b_i\right)$$

We can find a global bound in the same way as in Section 2.2 and obtain the same linearity results as in Section 2.3. To obtain the expected value of the MSE, we proceed as in Section 2.4 and obtain:

PROPOSITION 2.18.

$$E(\text{MSE}) \leq \frac{\Phi_T^2}{A_T A_{\min}} \frac{R_{\max}^2(1 + R_{\max})}{(1 - R_{\max})^2}$$

2.6 The Relation Between the $\frac{\Phi_T}{1-R_i}$, $\frac{\Phi_T}{R_i}$, and Φ_T

Estimators

It is interesting to study the relation between the three estimators defined in Sections 2.1, 2.5.1, and 2.5.2. The first estimator only scores a patch where the path dies, the second scores all the patches in the trajectory except where it dies, and the third scores all the patches in the trajectory. This gives a strong intuitive reason to consider the latter estimator as the best of all three. In this section we will give a mathematical support to the intuition. We have the relation among the three estimators:

$$\hat{\phi}_i'' = R_i\hat{\phi}_i' + (1 - R_i)\hat{\phi}_i$$

This implies that we should have the following relation between the variances:

$$\begin{aligned} \text{Var}(\hat{\phi}_i'') &= R_i^2 \text{Var}(\hat{\phi}_i') + (1 - R_i)^2 \text{Var}(\hat{\phi}_i) \\ &\quad + 2R_i(1 - R_i)\text{Cov}(\hat{\phi}_i', \hat{\phi}_i) \end{aligned} \quad (8)$$

The only value we don't know from the above expression

is the covariance. Once found, substituting each variance and the covariance with their value, we should obtain an identity.

PROPOSITION 2.19. The covariance between the $\hat{\phi}_i'$ and $\hat{\phi}_i$ estimators is given by

$$\text{Cov}(\hat{\phi}_i', \hat{\phi}_i) = \Phi_T \xi_i \phi_i - \phi_i^2$$

PROOF. We have

$$\text{Cov}(\hat{\phi}_i', \hat{\phi}_i) = E(\hat{\phi}_i'\hat{\phi}_i) - E(\hat{\phi}_i')E(\hat{\phi}_i) = E(\hat{\phi}_i'\hat{\phi}_i) - \phi_i^2$$

But

$$\begin{aligned} E(\hat{\phi}_i'\hat{\phi}_i) &= E\left(\left(\hat{\phi}_i^{(1)} + \hat{\phi}_i^{(2)} + \dots\right)\left(\hat{\phi}_i^{(1)} + \hat{\phi}_i^{(2)} + \dots\right)\right) \\ &= \sum_{1 \leq n < m} E\left(\hat{\phi}_i^{(n)}\hat{\phi}_i^{(m)}\right) + \sum_{1 \leq n \leq m} E\left(\hat{\phi}_i^{(n)}\hat{\phi}_i^{(m)}\right) \\ &= \sum_{1 \leq n < m} E\left(\hat{\phi}_i^{(n)}\hat{\phi}_i^{(m)}\right) \end{aligned} \quad (9)$$

The last equality follows because after a particle dies on a patch at length n , it can not survive on a patch at length $m \geq n$. That is

$$\sum_{1 \leq n \leq m} E\left(\hat{\phi}_i^{(n)}\hat{\phi}_i^{(m)}\right) = 0$$

Now we will obtain the last sum of (9). First we have:

$$\begin{aligned} E\left(\hat{\phi}_i^{(n)}\hat{\phi}_i^{(m)}\right) &= \sum_s \sum_{h_1, \dots, h_{n-1}} \sum_{h_{n+1}, \dots, h_{m-1}} \frac{\Phi_T}{R_i} \frac{\Phi_T}{(1 - R_i)} \\ &\quad \frac{\Phi_T^s}{\Phi_T} F_{sh_1} R_{h_1} \dots F_{h_{n-1}i} R_i \cdot \\ &\quad F_{ih_{n+1}} R_{h_{n+1}} \dots F_{h_{m-1}i} (1 - R_i) \\ &= \phi_i^{(n)} \Phi_T \sum_{h_{n+1}, \dots, h_{m-1}} F_{ih_{n+1}} R_{h_{n+1}} \dots F_{h_{m-1}i} \\ &= \phi_i^{(n)} \Phi_T \xi_i^{(m-n)} \end{aligned}$$

And then

$$\begin{aligned} E(\hat{\phi}_i'\hat{\phi}_i) &= \sum_{1 \leq n} \left(\sum_{n < m} E\left(\hat{\phi}_i^{(n)}\hat{\phi}_i^{(m)}\right) \right) \\ &= \sum_{1 \leq n} \phi_i^{(n)} \Phi_T \sum_{n < m} \xi_i^{(m-n)} \\ &= \Phi_T \xi_i \sum_{1 \leq n} \phi_i^{(n)} = \Phi_T \xi_i \phi_i \end{aligned}$$

□

It is easy to check that substituting the obtained value for the covariance and the three values for the respective variances in (8), we obtain an identity.

We will now compare the estimators. Comparing the respective variances, we easily obtain when the estimator $\frac{\Phi_T}{1-R_i}$ is better than the estimator $\frac{\Phi_T}{R_i}$, that is

PROPOSITION 2.20. For $R_i \leq \frac{1}{2}$ we have

$$\text{Var}(\hat{\phi}_i) \leq \text{Var}(\hat{\phi}'_i)$$

PROOF. It follows from the inequality

$$\frac{1}{1-R_i} \leq \frac{1}{R_i} + 2\xi_i$$

which holds for $R_i \leq \frac{1}{2}$, because $\xi_i \geq 0$. \square

Equally the estimator Φ_T is *always* better than the estimator $\frac{\Phi_T}{R_i}$:

PROPOSITION 2.21. *The inequality*

$$\text{Var}(\hat{\phi}''_i) \leq \text{Var}(\hat{\phi}'_i)$$

always holds.

PROOF. It follows from the inequality

$$1 + 2\xi_i \leq \frac{1}{R_i} + 2\xi_i$$

which holds for $R_i \leq 1$. \square

As the estimator Φ_T is a linear combination of the estimators $\frac{\Phi_T}{1-R_i}$ and $\frac{\Phi_T}{R_i}$, we can ask whether this combination is optimal. The answer is affirmative if we consider only direct illumination, that is for the case

$$\hat{\phi}''_i^{(1)} = R_i \hat{\phi}'_i^{(1)} + (1-R_i) \hat{\phi}_i^{(1)}$$

the combination can be shown to be optimal. In the general case we have

$$\hat{\phi}''_i = \alpha \hat{\phi}'_i + (1-\alpha) \hat{\phi}_i \quad 0 \leq \alpha \leq 1$$

and we must minimize

$$\begin{aligned} \text{Var}(\hat{\phi}''_i) &= \alpha^2 \text{Var}(\hat{\phi}'_i) + (1-\alpha)^2 \text{Var}(\hat{\phi}_i) \\ &\quad + 2\alpha(1-\alpha) \text{Cov}(\hat{\phi}'_i, \hat{\phi}_i) \end{aligned}$$

The optimal value for α is

$$\alpha = \frac{\text{Var}(\hat{\phi}_i) - \text{Cov}(\hat{\phi}'_i, \hat{\phi}_i)}{\text{Var}(\hat{\phi}'_i) + \text{Var}(\hat{\phi}_i) - 2\text{Cov}(\hat{\phi}'_i, \hat{\phi}_i)}$$

and after substituting and simplifying we obtain

$$\alpha = R_i(1 - \xi_i(1 - R_i))$$

As ξ_i can be assumed small respective to 1 (see Section 2.5.1), the estimator Φ_T is an optimal combination, and therefore it has lower variance than both components, in particular than the estimator $\frac{\Phi_T}{1-R_i}$. So a good heuristic is to consider the Φ_T estimator as the best estimator of the three considered.

2.7 An Infinite Path Length Estimator

For the sake of completeness, we introduce here an unbiased infinite path length estimator.

The random variables $\hat{\phi}''_i^{(1)}$, $\hat{\phi}''_i^{(2)}$, $\hat{\phi}''_i^{(3)}$, ... are defined in the following way:

All of those random variables are initially null. If the path γ arrives at patch i at length l , and if $s, h_1, h_2, \dots, h_{l-1}, i$ is the trajectory the path has followed, then the value of $\hat{\phi}''_i^{(l)}$ is set to $R_{h_1} R_{h_2} \dots R_{h_{l-1}} \Phi_T$. Let us also define a new random variable $\hat{\phi}'''_i$ as:

$$\hat{\phi}'''_i = \hat{\phi}''_i^{(1)} + \hat{\phi}''_i^{(2)} + \hat{\phi}''_i^{(3)} + \dots$$

It can be easily shown that those estimators are unbiased. Lower and upper bounds for the variance of the radiosity estimator $\hat{B}'''_i = E_i + \frac{R_i}{A_i} \hat{\phi}'''_i$ are

$$\frac{R_i^2}{A_i^2} \left(\left(\sum_{1 \leq n} R_{\min}^{n-1} \phi_i^{(n)} \right) \Phi_T (1 + 2R_i \xi_i) - \phi_i^2 \right) \leq \text{Var}(\hat{B}'''_i)$$

and

$$\text{Var}(\hat{B}'''_i) \leq \frac{R_i^2}{A_i^2} \left(\left(\sum_{1 \leq n} R_{\max}^{n-1} \phi_i^{(n)} \right) \Phi_T (1 + 2R_i \xi_i) - \phi_i^2 \right)$$

As $R_{\max} < 1$, we have

$$\begin{aligned} \text{Var}(\hat{B}'''_i) &< \frac{R_i^2}{A_i^2} \left(\left(\sum_{1 \leq n} \phi_i^{(n)} \right) \Phi_T (1 + 2R_i \xi_i) - \phi_i^2 \right) \\ &= \frac{R_i^2}{A_i^2} \left(\phi_i \Phi_T (1 + R_i \xi_i) - \phi_i^2 \right) = \text{Var}(\hat{B}'''_i) \end{aligned}$$

So at equal cost, this new estimator should be the best. The problem is that one can not deduce that the estimator $\hat{\phi}'''_i$ is better than $\hat{\phi}''_i$ because $\hat{\phi}'''_i$ is an *infinite* length estimator. Neither is it usable, of course. What we do then is to cut off the trajectory of a path when it reaches a predetermined length, or alternatively in the following way: A path s, h_1, h_2, \dots, h_k is stopped when $R_{h_1} R_{h_2} \dots R_{h_k} \leq t$, where t is a pre-established threshold. Then, if t is small enough we can be confident that the variance of the resulting biased estimator is near enough the variance of the infinite length estimator. As for the bias error, we can consider this small percentage of undistributed energy acceptable (the variance accounts for the noise in the image, the bias for the undistributed energy), or use *Russian Roulette*, which consists simply of switching to the Φ_T estimator to distribute the remaining energy. A thorough study of the efficiencies of those biased estimators is beyond the scope of this paper.

To obtain a bound for the expected value of the MSE, we proceed as in Section 2.4 and arrive at:

$$E(\text{MSE}) \leq \frac{\Phi_T^2}{A_T A_{\min}} \frac{R_{\max}^2 (1 + R_{\max})}{(1 - R_{\max})(1 - R_{\max}^2)}$$

3 GATHERING RANDOM WALK

In this section, we study four gathering estimators, analogous to the ones shown in Section 2. Some derivations are omitted or only sketched, as they follow the same approach as in Section 2.

3.1 An Unbiased Random-Walk Estimator for the Radiosity: The $\frac{E_s}{1-R_s}$ Estimator

Let us first consider what the expected value of any unbiased Monte Carlo estimator should be for the radiosity of a patch. Let us suppose that the emittance of source s is E_s , b_i is the reflected radiosity, or radiosity of patch i due to the received power (that is $b_i = B_i - E_i$, thus for a nonemitter patch it equals the total radiosity), F_{kl} denotes the Form-Factor from patch k to patch l , and R_k denotes the reflectance of patch k . Then we have, by developing the Radiosity system in Neumann series (dropping the zero order term):

$$b_i = R_i \sum_s E_s F_{is} + R_i \sum_h \sum_s E_s F_{ih} R_h F_{hs} + R_i \sum_h \sum_j \sum_s E_s F_{ih} R_h F_{hj} R_j F_{js} + \dots$$

This can be expressed as:

$$b_i = b_i^{(1)} + b_i^{(2)} + b_i^{(3)} + \dots$$

where

$$b_i^{(1)} = R_i \sum_s E_s F_{is}, \quad b_i^{(2)} = R_i \sum_s \sum_h E_s F_{ih} R_h F_{hs}, \\ b_i^{(3)} = R_i \sum_s \sum_h \sum_j E_s F_{ih} R_h F_{hj} R_j F_{js},$$

and so on. That is, $b_i^{(1)}$ represents the radiosity due to direct illumination, $b_i^{(2)}$ represents the radiosity after one bounce, and so on. It is also useful to define the following quantities:

$$b_{is} = b_{is}^{(1)} + b_{is}^{(2)} + \dots$$

$b_{is}^{(1)}$ represents the radiosity due to direct illumination from source s , $b_{is}^{(2)}$ represents the radiosity after one bounce from source s , and so on. It is clear that:

$$b_i = \sum_s b_{is}$$

Let us now consider the following simulation. A path starts from patch i with probability p_i (this probability can be considered as the initial or received importance of the patch), and goes to patch j with probability F_{ij} . Then it survives or dies according to the probabilities $(R_j, 1 - R_j)$. The expected length of the trajectory is bound by $\frac{1}{1-R_{max}}$ [17], where R_{max} is the maximum of the reflectivities. Now let us define for patch i and path γ the family of random variables $\hat{b}_i^{(1)}, \hat{b}_i^{(2)}, \hat{b}_i^{(3)}, \dots$ in the following way:

All of those random variables are initially null. If the path γ happens to die on source s at length l , then the value of $\hat{b}_i^{(l)}$ is set to $R_i \frac{E_s}{p_i(1-R_s)}$. Let us also define a new random variable \hat{b}_i as:

$$\hat{b}_i = \hat{b}_i^{(1)} + \hat{b}_i^{(2)} + \hat{b}_i^{(3)} + \dots$$

PROPOSITION 3.1. *The random variable $\hat{b}_i^{(l)}$ is an unbiased estimator for the radiosity due to the power arrived at patch i*

after l bounces (reflected radiosity), and \hat{b}_i is an unbiased estimator for the total radiosity of patch i due to the power arrived after any number of bounces.

PROOF. Applying the definition of expected value, and remembering that the probability of selecting patch i is p_i , the probability of arriving at source s just after leaving patch i is F_{is} and the probability of dying on source s is $1 - R_s$, we have

$$E(\hat{b}_i^{(1)}) = \sum_s R_i \frac{E_s}{p_i(1-R_s)} p_i F_{is} (1-R_s) = b_i^{(1)}$$

Now, to go from patch i to a source s in a two length path we can pass through any patch h , and survive in it with probability R_h , so we have

$$E(\hat{b}_i^{(2)}) = \sum_h \sum_s R_i \frac{E_s}{p_i(1-R_s)} p_i F_{ih} R_h F_{hs} (1-R_s) = b_i^{(2)}$$

and so on. Then, we have

$$E(\hat{b}_i) = E(\hat{b}_i^{(1)} + \hat{b}_i^{(2)} + \dots) = E(\hat{b}_i^{(1)}) + E(\hat{b}_i^{(2)}) + \dots \\ = b_i^{(1)} + b_i^{(2)} + \dots = b_i$$

□

PROPOSITION 3.2. *The variance of the estimator \hat{b}_i is given by*

$$\text{Var}(\hat{b}_i) = \frac{R_i}{p_i} \sum_s \frac{E_s}{(1-R_s)} b_{is} - b_i^2$$

PROOF. We can use an analogous decomposition for the variance to that given by (1) and observing that each term of the form $E(\hat{b}_i^{(n)} \hat{b}_i^{(m)})$ is null, because if a path dies at length n on a certain source it cannot die again at length m , we have

$$\text{Var}(\hat{b}_i) = E(\hat{b}_i^{(1)2}) + E(\hat{b}_i^{(2)2}) + \dots - b_i^2$$

But

$$E(\hat{b}_i^{(1)2}) = \sum_s \left(R_i \frac{E_s}{p_i(1-R_s)} \right)^2 p_i F_{is} (1-R_s) \\ = \sum_s R_i \frac{E_s}{p_i(1-R_s)} b_{is}^{(1)} \\ E(\hat{b}_i^{(2)2}) = \sum_h \sum_s \left(R_i \frac{E_s}{p_i(1-R_s)} \right)^2 p_i F_{ih} R_h F_{hs} (1-R_s) \\ = \sum_s R_i \frac{E_s}{p_i(1-R_s)} b_{is}^{(2)}$$

and so on. Then we obtain

$$\text{Var}(\hat{b}_i) = \sum_s R_i \frac{E_s}{p_i(1-R_s)} (b_{is}^{(1)} + b_{is}^{(2)} + \dots) - b_i^2 \\ = \frac{R_i}{p_i} \sum_s \frac{E_s}{(1-R_s)} b_{is} - b_i^2$$

□

For the radiosity, our estimator is simply $\hat{b}_i + E_i$. So, as E_i

is a constant, we have

$$\text{Var}(\hat{b}_i + E_i) = \text{Var}(\hat{b}_i) = \frac{R_i}{p_i} \sum_s \frac{E_s}{(1-R_s)} b_{is} - b_i^2 \quad (10)$$

3.2 Some Particular Cases

Suppose we are only interested in patch i . We put $p_i = 1$, and then

$$\text{Var}(\hat{b}_i) = R_i \sum_s \frac{E_s}{(1-R_s)} b_{is} - b_i^2 \quad (11)$$

A remarkable property of (11) is that it is also valid for the *nondiscrete* case. This is because (11) is independent of any discretization of the scene, and so it gives us the variance for the estimator of the radiosity of the *point* origin of the path (it is not possible to obtain formulae for the nondiscrete shooting case because the probability of a path impinging on a given point is null). For a single source, $b_{is} = b_i$, and if we trace N paths:

$$\text{Var}(\hat{b}_i) = \frac{1}{N} \left(\frac{R_i E_s}{(1-R_s)} b_i - b_i^2 \right)$$

If we are interested in all the patches without preferring any one in particular, a possibility is to consider $p_i = \frac{A_i}{A_T}$,

where A_i is the area of patch i and A_T is the total area. This means assigning importance to them according to their area. We have, for N paths

$$\text{Var}(\hat{b}_i) = \frac{1}{N} \left(\frac{R_i A_T}{A_i} \sum_s \frac{E_s}{(1-R_s)} b_{is} - b_i^2 \right)$$

3.3 A Global Bound for All the Variances

Let us see how we can find a bound for all variances.

For all patches i , we have, remembering that $b_i \leq R_i \mathcal{B}$ and $\frac{E_s}{1-R_s} \leq \mathcal{B}$ (see Appendix), and calling A_{min} the minimum area:

$$\begin{aligned} \text{Var}(\hat{b}_i) &= \frac{R_i}{p_i} \sum_s \frac{E_s}{(1-R_s)} b_{is} - b_i^2 \leq \frac{R_i \mathcal{B}}{p_i} \sum_s b_{is} \\ &\leq \frac{R_i \mathcal{B}}{p_i} b_i \leq \frac{R_{max}^2 \mathcal{B}^2}{p_{min}} \end{aligned}$$

and for N paths

$$\text{Var}(\hat{b}_i) \leq \frac{R_{max}^2 \mathcal{B}^2}{N p_{min}}$$

When $p_i = \frac{A_i}{A_T}$ we obtain

$$\text{Var}(\hat{b}_i) \leq \frac{R_{max}^2 A_T \mathcal{B}^2}{N A_{min}}$$

This means that we can always obtain a number of paths N so that the variance for any patch is below any pre-established threshold. Observe also that this bound keeps holding when we add emissivity to any patch j under the constraint $\frac{E_j}{1-R_j} \leq \mathcal{B}$.

3.4 Complexity

From the previous section, given a bound V for all variances we can find the number of paths N to fulfill this bound:

$$N = V^{-1} A_T A_{min}^{-1} R_{max}^2 \mathcal{B}^2$$

And so, remembering from Section 2.3 the cost C_1 for a path, the total cost C_T of the N paths is given by:

- 1) structured scene, bound cost for picking a patch within a surface

$$C_T = O\left(V^{-1} A_T A_{min}^{-1} R_{max}^2 \mathcal{B}^2 \log n_s\right)$$

- 2) structured scene, hierarchical structure of patches within a surface

$$C_T = O\left(V^{-1} A_T A_{min}^{-1} R_{max}^2 \mathcal{B}^2 \max\left(\log n_s, \log \frac{n_p}{n_s}\right)\right)$$

- 3) nonstructured scene, bound cost for picking a patch within a surface

$$C_T = O\left(V^{-1} A_T A_{min}^{-1} R_{max}^2 \mathcal{B}^2 n_s\right)$$

- 4) nonstructured scene, hierarchical structure of patches within a surface

$$C_T = O\left(V^{-1} A_T A_{min}^{-1} R_{max}^2 \mathcal{B}^2 \max\left(n_s, \log \frac{n_p}{n_s}\right)\right)$$

Introducing the same scenarios as in Section 2.3 and following the same discussion, taking into account that in scenario 1, $O(A_T) = O(n_s)$ and in scenario 2 $O(A_T) = O(1)$, we obtain the results for gathering in Table 2. We can compare them in the same table with the results for shooting. For scenario 1 the complexity is higher in the gathering case. This is because adding new area means adding new sources of paths (or sources of importance). This was not allowed in the shooting case, that is, we kept the total power constant.

TABLE 2
COMPLEXITY FOR THE DIFFERENT CASES AND SCENARIOS FOR SHOOTING AND GATHERING RANDOM WALK.

Shooting	1	2
1	$O(\log n_p)$	$O(n_p)$
2	$O(\log n_p)$	$O(n_p \log n_p)$
3	$O(n_p)$	$O(n_p)$
4	$O(n_p)$	$O(n_p \log n_p)$
Gathering	1	2
1	$O(n_p \log n_p)$	$O(n_p)$
2	$O(n_p \log n_p)$	$O(n_p \log n_p)$
3	$O(n_p^2)$	$O(n_p)$
4	$O(n_p^2)$	$O(n_p \log n_p)$

3.5 Expected Value of the Mean Square Error

In this section we will bound the expected value of the MSE when $p_i = \frac{A_i}{A_T}$. Let us first bound $\sum_i b_{is}$. Using (3), we can easily prove

PROPOSITION 3.3.

$$\sum_s b_{is} \leq \frac{R_{\max} \Phi_s}{A_{\min}(1 - R_{\max})}$$

Then we have for the expected value of the MSE:

PROPOSITION 3.4.

$$E(\text{MSE}) \leq \frac{R_{\max}^2 \mathcal{B} \Phi_T}{A_{\min}(1 - R_{\max})}$$

PROOF.

$$\begin{aligned} E(\text{MSE}) &= \frac{1}{A_T} \sum_i A_i \text{Var}(\hat{b}_i) \\ &= \frac{1}{A_T} \sum_i A_i \left(\frac{R_i A_T}{A_i} \sum_s \frac{E_s}{(1 - R_s)} b_{is} - b_i^2 \right) \\ &\leq R_{\max} \sum_i \sum_s \frac{E_s}{(1 - R_s)} b_{is} \\ &\leq R_{\max} \sum_s \frac{E_s}{(1 - R_s)} \frac{R_{\max} \Phi_s}{A_{\min}(1 - R_{\max})} \\ &\leq \frac{R_{\max}^2 \mathcal{B}}{A_{\min}(1 - R_{\max})} \sum_s \Phi_s = \frac{R_{\max}^2 \mathcal{B} \Phi_T}{A_{\min}(1 - R_{\max})} \end{aligned}$$

□

And for N paths

$$E(\text{MSE}) \leq \frac{R_{\max}^2 \mathcal{B} \Phi_T}{N A_{\min}(1 - R_{\max})} \quad (12)$$

Interestingly, if we take the expected value of the MSE as a measure of the error upon which to study the complexity, from (12) we obtain linearity, following the same discussion as in Section 2.3. This result is not contradictory to the one in Section 3.4, because there we took a different measure, the individual variances of the patches.

3.6 Other Unbiased Estimators

3.6.1 The $\frac{E_s}{R_s}$ Estimator

Let us now define for patch i and path γ another family of random variables $\hat{b}_i^{(1)}, \hat{b}_i^{(2)}, \hat{b}_i^{(3)}, \dots$ in the following way:

All of those random variables are initially null. If the path γ happens to survive on source s at length l , then the value of $\hat{b}_i^{(l)}$ is set to $R_i \frac{E_s}{p_i R_s}$. Let us also define a new random variable \hat{b}'_i as:

$$\hat{b}'_i = \hat{b}_i^{(1)} + \hat{b}_i^{(2)} + \hat{b}_i^{(3)} + \dots$$

We can proceed as in Section 3.1 and prove that our estimators are unbiased. As for the variance we have

PROPOSITION 3.5.

$$\text{Var}(\hat{b}'_i) = \frac{R_i}{p_i} \sum_s \frac{E_s + 2b_s}{R_s} b_{is} - b_i^2$$

PROOF. We can decompose $\text{Var}(\hat{b}'_i)$ as in (1), but now the terms of the form $E(\hat{b}_i^{(n)} \hat{b}_i^{(m)})$ are no longer null, because if a path survives at length n on source s it can also survive on source s' at length m . We must then obtain the value of those terms. Proceeding as in the proof of Proposition 2.10 we obtain

$$E(\hat{b}_i^{(n)} \hat{b}_i^{(m)}) = \frac{R_i}{p_i} \sum_s b_{is}^{(n)} \frac{b_s^{(m-n)}}{R_s}$$

Then

$$\begin{aligned} \sum_{1 \leq n < m} \left(\sum E(\hat{b}_i^{(n)} \hat{b}_i^{(m)}) \right) &= \frac{R_i}{p_i} \sum_s \sum_{1 \leq n} b_{is}^{(n)} \sum_{n \leq m} \frac{b_s^{(m-n)}}{R_s} \\ &= \frac{R_i}{p_i} \sum_s b_{is} \frac{b_s}{R_s} \end{aligned}$$

and also

$$\begin{aligned} E(\hat{b}_i^{(1)2}) &= \sum_s \left(R_i \frac{E_s}{p_i R_s} \right)^2 p_i F_{is} R_s = \sum_s R_i \frac{E_s}{p_i R_s} b_{is}^{(1)} \\ E(\hat{b}_i^{(2)2}) &= \sum_h \sum_s \left(R_i \frac{E_s}{p_i R_s} \right)^2 p_i F_{ih} R_h F_{hs} R_s \\ &= \sum_s R_i \frac{E_s}{p_i R_s} b_{is}^{(2)} \end{aligned}$$

and so on. Then we obtain

$$\begin{aligned} \text{Var}(\hat{b}'_i) &= \frac{R_i}{p_i} \sum_s \frac{E_s}{R_s} (b_{is}^{(1)} + b_{is}^{(2)} + \dots) \\ &\quad + 2 \frac{R_i}{p_i} \sum_s b_{is} \frac{b_s}{R_s} - b_i^2 \\ &= \frac{R_i}{p_i} \sum_s \frac{E_s + 2b_s}{R_s} b_{is} - b_i^2 \end{aligned}$$

□

For the radiosity our estimator is simply $\hat{b}'_i + E_i$, and as E_i is a constant we have

$$\text{Var}(\hat{b}'_i + E_i) = \text{Var}(\hat{b}'_i) = \frac{R_i}{p_i} \sum_s \frac{E_s + 2b_s}{R_s} b_{is} - b_i^2 \quad (13)$$

And this variance can be bound as in Section 3.3. Putting $p_i = 1$ or $p_i = \frac{A_i}{A_T}$ we have the same particular cases as in Section 3.2. Now, to obtain the expected value of the MSE when $p_i = \frac{A_i}{A_T}$ we proceed as in Section 3.5 and arrive at:

$$E(\text{MSE}) \leq \frac{R_{\max}^2}{A_{\min}(1 - R_{\max})} \sum_s \frac{\Phi_s (E_s + 2R_s \mathcal{B})}{R_s}$$

3.6.2 The E_s Estimator

Let us now define for patch i and path γ another family of random variables $\hat{b}_i^{\prime(1)}, \hat{b}_i^{\prime(2)}, \hat{b}_i^{\prime(3)}, \dots$ in the following way:

All of those random variables are initially null. If the path γ happens to arrive at source s at length l , then the value of $\hat{b}_i^{\prime(l)}$ is set to $R_l \frac{E_s}{p_i}$. Let us also define a new random variable $\hat{b}_i^{\prime\prime}$ as:

$$\hat{b}_i^{\prime\prime} = \hat{b}_i^{\prime(1)} + \hat{b}_i^{\prime(2)} + \hat{b}_i^{\prime(3)} + \dots$$

We can easily prove as in Section 3.1 that our estimators are unbiased. As for the variance we obtain, proceeding as in Section 3.6.1.

PROPOSITION 3.6.

$$\text{Var}(\hat{b}_i^{\prime\prime}) = \frac{R_l}{p_i} \sum_s (E_s + 2b_s) b_{is} - b_i^2$$

For the radiosity our estimator is simply $\hat{b}_i^{\prime\prime} + E_i$, and as E_i is a constant we have

$$\text{Var}(\hat{b}_i^{\prime\prime} + E_i) = \text{Var}(\hat{b}_i^{\prime\prime}) = \frac{R_l}{p_i} \sum_s (E_s + 2b_s) b_{is} - b_i^2 \quad (14)$$

And this variance can be bound as in Section 3.3. Putting $p_i = 1$ or $p_i = \frac{A_i}{A_T}$ we have the same particular cases as in Section 3.2. Now, to obtain the expected value of the MSE when $p_i = \frac{A_i}{A_T}$ we proceed as in Section 3.5 and arrive at:

$$E(\text{MSE}) \leq \frac{R_{max}^2}{A_{min}(1 - R_{max})} \sum_s \Phi_s (E_s + 2R_s \mathcal{B})$$

3.7 The Relation Between the $\frac{E_s}{1-R_s}$, $\frac{E_s}{R_s}$ and E_s

Estimators

It is interesting to study the relation between the three estimators defined in Sections 3.1, 3.6.1, and 3.6.2. If we use the following decomposition

$$\hat{b}_i = \sum_s \hat{b}_{is}$$

where \hat{b}_{is} is the estimator for the radiosity due to source s , and similarly for the other estimators, we have the relation

$$\hat{b}_i^{\prime\prime} = \sum_s R_s \hat{b}_{is} + (1 - R_s) \hat{b}_{is} \quad (15)$$

For the particular case where all sources have the same reflectivity R (this is obviously the case for a single source), we have simply

$$\hat{b}_i^{\prime\prime} = R \hat{b}_i^{\prime} + (1 - R) \hat{b}_i$$

This implies that we should have the following relation between the variances:

$$\begin{aligned} \text{Var}(\hat{b}_i^{\prime\prime}) &= R^2 \text{Var}(\hat{b}_i^{\prime}) + (1 - R)^2 \text{Var}(\hat{b}_i) \\ &\quad + 2R(1 - R) \text{Cov}(\hat{b}_i^{\prime}, \hat{b}_i) \end{aligned} \quad (16)$$

The only value we don't know from the above expression is the covariance. This can be found proceeding as in

Proposition 2.19, and we obtain

$$\text{Cov}(\hat{b}_i^{\prime}, \hat{b}_i) = \frac{R_l}{p_i} \sum_s b_{is} \frac{b_s}{R} - b_i^2$$

We can easily check that substituting the obtained value and the three values for the respective variances (considering all sources with the same reflectivity R) in (16) we obtain an identity.

Another interesting point is the comparison of the estimators. From the respective formulae for the variances we can see that the estimator $\frac{E_s}{1-R_s}$ is better than the estimator $\frac{E_s}{R_s}$, that is

$$\text{Var}(\hat{b}_i) \leq \text{Var}(\hat{b}_i^{\prime})$$

when $R_s \leq \frac{1}{2}$ for all sources s .

Equally as

$$\text{Var}(\hat{b}_i^{\prime}) \leq \text{Var}(\hat{b}_i^{\prime\prime})$$

is always true, the estimator E_s is *always* better than the estimator $\frac{E_s}{R_s}$.

For the particular case considered above where all sources have the same reflectivity R , as the estimator E_s is a linear combination of the estimators $\frac{E_s}{1-R_s}$ and $\frac{E_s}{R_s}$, we can ask whether this combination is optimal. The answer is affirmative if we consider only direct illumination, that is for the case

$$\hat{b}_i^{\prime(1)} = R \hat{b}_i^{\prime(1)} + (1 - R) \hat{b}_i^{\prime(1)}$$

the combination can be shown to be optimal. In the general case we have

$$\hat{b}_i^{\prime\prime} = \alpha \hat{b}_i^{\prime} + (1 - \alpha) \hat{b}_i \quad 0 \leq \alpha \leq 1$$

and minimizing the variance we obtain as optimal value

$$\alpha = R - (1 - R) \frac{\sum_s b_s b_{is}}{\sum_s E_s b_{is}}$$

If for all s , b_s can be assumed small respective to E_s , the estimator E_s is an optimal combination, and therefore, it has a lower variance than both components. So a good heuristic is to consider the E_s estimator as the best estimator of the three considered.

That the general case (15) is also heuristically optimal can be seen because each term in the sum represents an optimal combination for a single source case.

3.8 An Infinite Path Length Estimator

For the sake of completeness, we introduce here an unbiased infinite path length estimator.

The random variables $\hat{b}_i^{\prime\prime(1)}, \hat{b}_i^{\prime\prime(2)}, \hat{b}_i^{\prime\prime(3)}, \dots$ are defined in the following way:

All of those random variables are initially null. If the path γ happens to arrive at source s at length l , and if $i, h_1, h_2, \dots, h_{l-1}, s$ is the trajectory the path has followed, then the

value of $\hat{b}_i^{(l)}$ is set to $R_i R_{h_1} R_{h_2} \dots R_{h_{l-1}} \frac{E_s}{p_i}$. Let us also define a new random variable \hat{b}_i''' as:

$$\hat{b}_i''' = \hat{b}_i^{(1)} + \hat{b}_i^{(2)} + \hat{b}_i^{(3)} + \dots$$

It can be easily shown that those estimators are unbiased. Lower and upper bounds for the variance of the radiosity estimator, $\hat{b}_i'' + E_i$, are given by

$$\frac{R_i}{p_i} \sum_s (E_s + 2b_s) \sum_n R_{\min}^{n-1} b_{is}^{(n)} - b_i^2 \leq \text{Var}(\hat{b}_i'' + E_i)$$

and

$$\text{Var}(\hat{b}_i''' + E_i) \leq \frac{R_i}{p_i} \sum_s (E_s + 2b_s) \sum_n R_{\max}^{n-1} b_{is}^{(n)} - b_i^2$$

As $R_{\max} < 1$, we have

$$\begin{aligned} \text{Var}(\hat{b}_i''' + E_i) &< \frac{R_i}{p_i} \sum_s (E_s + 2b_s) \sum_n b_{is}^{(n)} - b_i^2 \\ &= \frac{R_i}{p_i} \sum_s (E_s + 2b_s) b_{is} - b_i^2 = \text{Var}(\hat{b}_i' + E_i) \end{aligned}$$

Now the same discussion would follow as in Section 2.8.

To obtain a bound for the expected value of the MSE, when $p_i = \frac{A_i}{A_T}$, we proceed as in Section 3.5 and taking into account that

$$\sum_i b_{is}^{(n)} \leq \frac{R_{\max}}{A_{\min}} \sum_i \phi_{is}^{(n)} \leq \frac{R_{\max}}{A_{\min}} \Phi_s$$

we arrive at:

$$E(\text{MSE}) \leq \frac{R_{\max}^2}{A_{\min}(1 - R_{\max}^2)} \sum_s \Phi_s (E_s + 2R_s \mathcal{B})$$

4 RESULTS

4.1 Shooting Monte Carlo

Here we present some experiments performed on a very simple scene, a cubical enclosure with each face divided into nine equal size patches, the reflectivities of the faces being 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, respectively, and a source with emissivity 1 in the middle of the first face, in patch 4. Thus patches 1 to 9 receive no direct lighting and have reflectivity 0.3, patches 10 to 18 reflectivity 0.4, and so on. The disposition of the patches is shown in Fig. 1. For this scene we computed a reference solution with the $\frac{\Phi_T}{1-R_i}$ estimator and 10^8 paths. With the radiosity values so obtained we computed the variances (for a single path) with the formulae for the variances obtained in Section 2 for the three estimators: $\frac{\Phi_T}{1-R_i}$ (2), $\frac{\Phi_T}{R_i}$ (6), and Φ_T (7), with the approximation $\xi_i = 0$. After that, for each estimator we ran 100 executions of 10^4 paths each, to obtain 100 sets of radiosities, that were compared to the reference solution to obtain the square errors. Those 100 sets of square errors were averaged and so we obtained an estimator of the variance for one single path (after multiplying by 10^4). In Figs. 2a, 2b,

and 2c, respectively, we compare the theoretically expected variances (square dots), and the experimentally obtained ones (average of square errors) for the $\frac{\Phi_T}{1-R_i}$ (diamonds), $\frac{\Phi_T}{R_i}$ (triangles), and Φ_T (circles) estimators, respectively. The big differences between the variances for the different estimators account for the differences between the vertical axes. In Fig. 2d we compare the experimentally obtained variances (average of square errors) for the three estimators used. The results are as expected from the theoretical findings in Section 2.6. The estimator Φ_T shows itself as the best for all patches, while the $\frac{\Phi_T}{1-R_i}$ estimator is better than the $\frac{\Phi_T}{R_i}$ estimator for patches with reflectivity less than 0.5, that is, patches from 1 to 27. It must be remarked here that in the respective formulae for the variances the fact that the radiosity of a patch is due to direct or indirect illumination is completely irrelevant; but directly illuminated patches usually receive more radiosity than the other ones, and the variances, being inverted parabola functions, can be considered within a wide interval to be increasing with respect to reflected radiosity. Thus a higher variance is to be expected with directly illuminated patches.

4.2 Gathering Monte Carlo

With the same scene and reference radiosities as in the previous section, we computed the variances (for one path) with the formulae for the variances obtained in Section 3 for the three estimators: $\frac{E_s}{1-R_s}$ (10), $\frac{E_s}{R_s}$ (13), and E_s (14), taking $p_i = \frac{A_i}{A_T}$, and with the approximation $b_s = 0$. After that, for each estimator we ran 100 executions of 10^4 paths each, to obtain 100 sets of radiosities, that were compared to the reference solution to obtain the square errors. Those 100 sets of square errors were averaged and so we obtained an estimator of the variance for one path (after multiplying by 10^4). In Figs. 3a, 3b, and 3c, respectively, we compare the theoretically expected variances (square dots), and the experimentally obtained ones (average of square errors) for the three estimators: the $\frac{E_s}{1-R_s}$ (diamonds), $\frac{E_s}{R_s}$ (triangles) and E_s (circles). In Fig. 3d we give the experimentally obtained variances (average of square errors) for the three estimators used. The results are as expected from the theoretical findings in Section 3.7. The estimator E_s shows itself as the best for all patches, while the $\frac{E_s}{1-R_s}$ estimator is better than the $\frac{E_s}{R_s}$ estimator because the source reflectivity (0.3) is less than 0.5. The variances are much higher than for the shooting method; this is because each path from the source in the shooting method updates every visited patch, while a path from a given patch in the gathering methods only updates this given patch. In addition to that a path can die without having hit any source, in this case its contribution is null.

5 CONCLUSIONS

In this paper we have studied the error and complexity of Random Walk Monte Carlo Radiosity, both shooting and gathering methods. The complexity is summarized in Table 2.

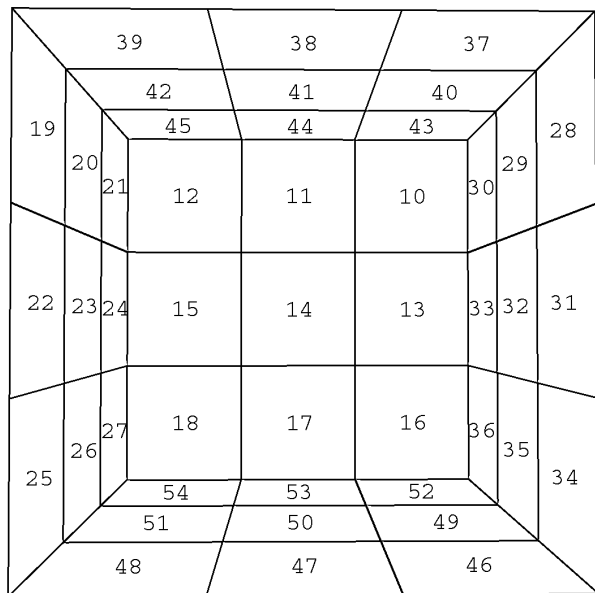


Fig. 1. Numbering the patches in the test scene. Patches 1 to 9, with reflectance 0.3, are not shown. Patch 4 is the emitter. Patches 10 to 18 have reflectance 0.4, 19 to 27, 0.5, and so on.

Given some constraints, the shooting random walk technique exhibits a linear complexity with respect to the number of patches. We also have obtained closed forms and bounds for the variances of three unbiased estimators for both shooting and gathering methods. We have studied the relative efficiency of the three estimators for each case, shooting and gathering. The variances of the different estimators are shown in Table 3. Bounds are also given for the infinite path length estimators, and for the expected values of the Mean Square Errors. The MSE bounds are shown in Table 4. Our future work in the subject will be, first, to study the efficiency of biased estimators, second, to study the RGB case, that is, to obtain the best estimator (unbiased or not) for a color scene. We will also try to obtain actual bounds for the discretization error based on the theoretical bounds given in [1]. Another subject of research will be to obtain heuristics, based on the variances, for hybrid methods using both shooting and gathering; for instance a first coarse pass for shooting, and a posterior refinement for gathering on small or more important patches. Pavol Elias suggested (personal communication) that the methods in [16] and [7] can be considered as a breadth-first approach to the Φ_T estimator, which in turn would represent a depth-first approach. In this case, complexity and variance results should also apply to [16] and [7]. If the same results apply to other Monte Carlo techniques will be investigated.

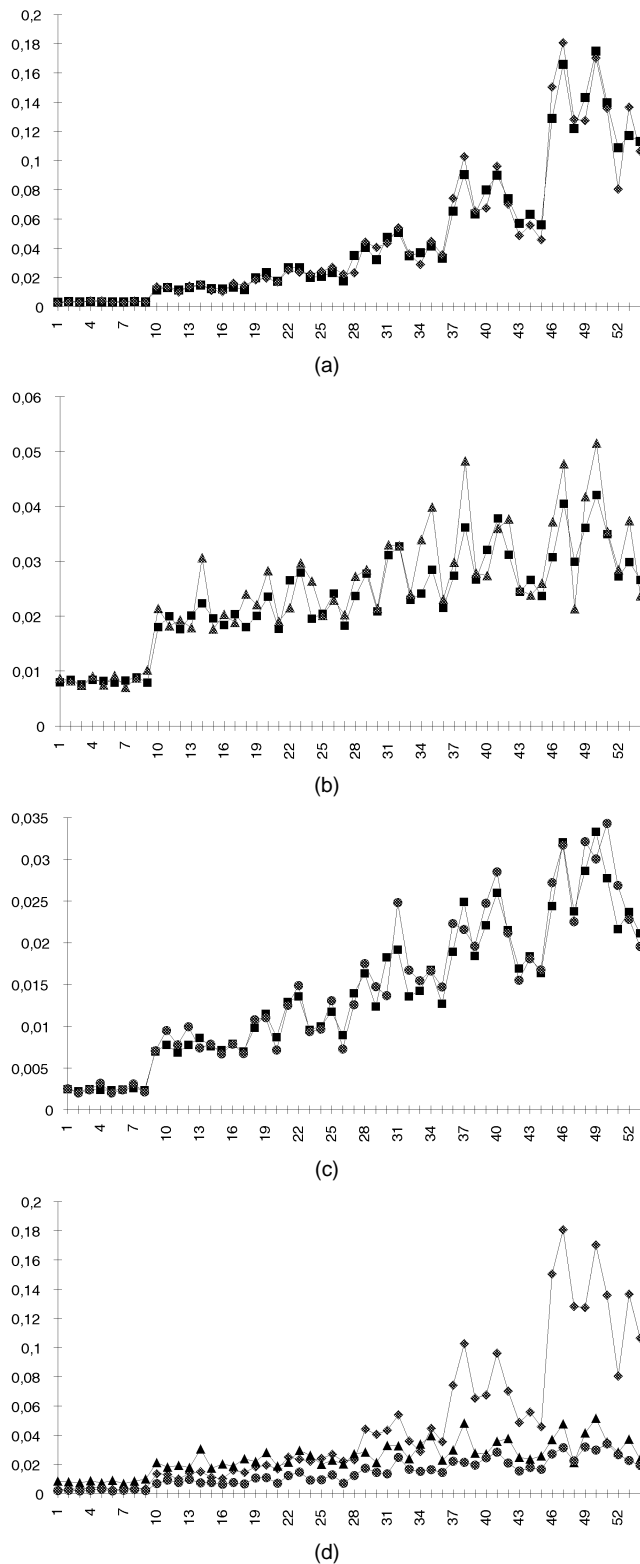


Fig. 2. Comparison of the expected variances (plotted as square dots) and the experimentally obtained square errors for the $\frac{\Phi_T}{1-R_i}$ (a, diamonds), $\frac{\Phi_T}{R_i}$ (b, triangles), and Φ_T (c, circles) estimators, for the 54 equal area patches of a cube (on x axis), with face reflectivities 0.3, 0.4, 0.5, 0.6, 0.7, 0.8. A source with emittance 1 is in the middle of the first face, in patch 4. Patches 1 to 9 receive no direct lighting. In (d), we compare the results for the different estimators.

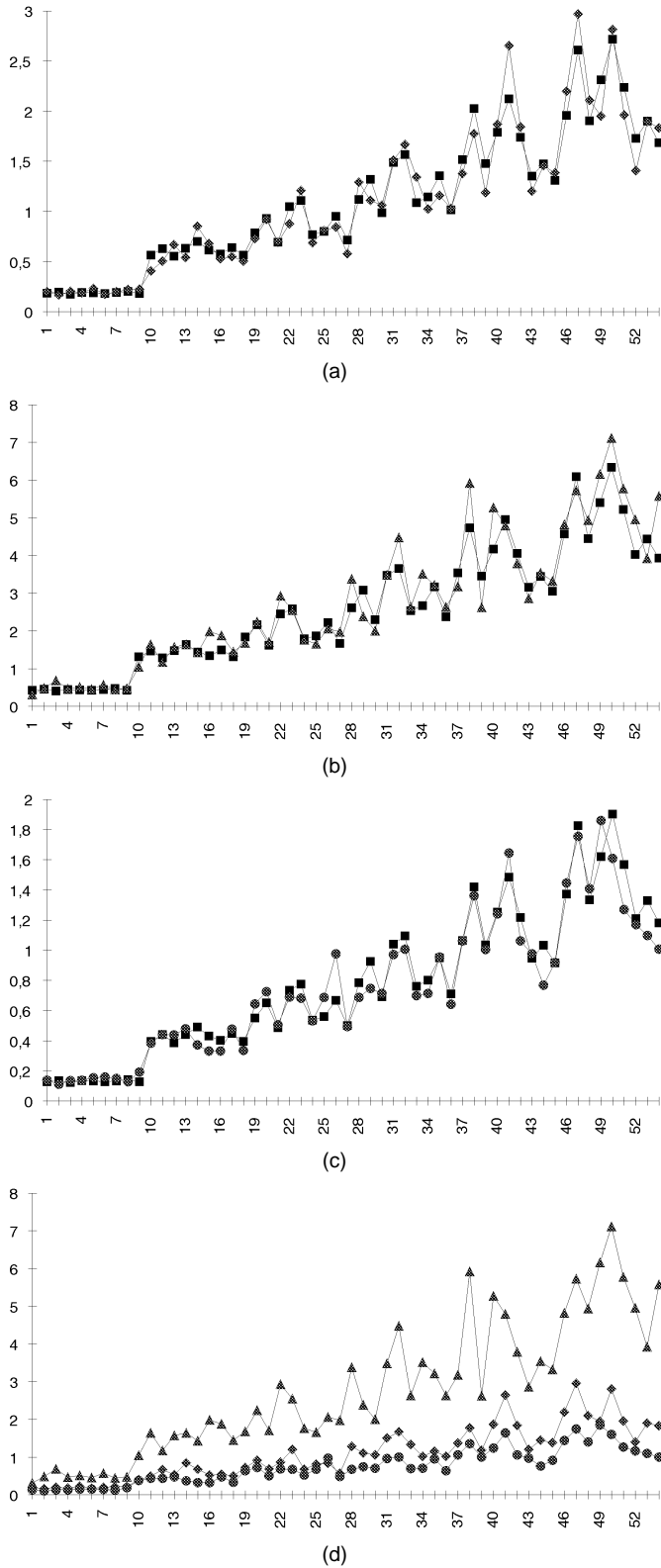


Fig. 3. Comparison of the expected variances (plotted as square dots) and the experimentally obtained square errors for the $\frac{E_s}{1-R_s}$ (a, diamonds), $\frac{E_s}{R_s}$ (b, triangles), and E_s (c, circles) estimators, for the same scene as Fig. 2. In (d), we compare the results for the different estimators.

TABLE 3
VARIANCES FOR THE DIFFERENT ESTIMATORS

Estimator	Variance
$\frac{\Phi_T}{1-R_i}$	$b_i \left(\frac{\Phi_T R_i}{A_i(1-R_i)} - b_i \right)$
$\frac{\Phi_T}{R_i}$	$b_i \left(\frac{\Phi_T R_i}{A_i} \left(\frac{1}{R_i} + 2\xi_i \right) - b_i \right)$
Φ_T	$b_i \left(\frac{R_i \Phi_T}{A_i} (1 + 2R_i \xi_i) - b_i \right)$
$\frac{E_s}{1-R_s}$	$\frac{R_i}{\rho_i} \sum_s \frac{E_s}{(1-R_s)} b_{is} - b_i^2$
$\frac{E_s}{R_s}$	$\frac{R_i}{\rho_i} \sum_s \frac{E_s + 2b_s}{R_s} b_{is} - b_i^2$
E_s	$\frac{R_i}{\rho_i} \sum_s (E_s + 2b_s) b_{is} - b_i^2$

TABLE 4
BOUNDS FOR THE EXPECTED VALUE OF THE MSE FOR THE DIFFERENT ESTIMATORS

Estimator	MSE
$\frac{\Phi_T}{1-R_i}$	$\frac{\Phi_T^2}{A_T A_{min}} \frac{R_{max}^2}{(1-R_{max})^2}$
$\frac{\Phi_T}{R_i}$	$\frac{\Phi_T^2}{A_T A_{min}} \frac{R_{max} (1+R_{max})}{(1-R_{max})^2}$
Φ_T	$\frac{\Phi_T^2}{A_T A_{min}} \frac{R_{max}^2 (1+R_{max})}{(1-R_{max})^2}$
shoot.inf.p.length	$\frac{\Phi_T^2}{A_T A_{min}} \frac{R_{max}^2 (1+R_{max})}{(1-R_{max})(1-R_{max}^2)}$
$\frac{E_s}{1-R_s}$	$\frac{R_{max}^2 \mathcal{B} \Phi_T}{A_{min} (1-R_{max})}$
$\frac{E_s}{R_s}$	$\frac{R_{max}^2}{A_{min} (1-R_{max})} \sum_s \frac{\Phi_s (E_s + 2R_s \mathcal{B})}{R_s}$
E_s	$\frac{R_{max}^2}{A_{min} (1-R_{max})} \sum_s \Phi_s (E_s + 2R_s \mathcal{B})$
gather.inf.p.length	$\frac{R_{max}^2}{A_{min} (1-R_{max}^2)} \sum_s \Phi_s (E_s + 2R_s \mathcal{B})$

APPENDIX

The n_p radiosities solution of the Radiosity system of equations

$$B_i = R_i \sum_{j=1}^{n_p} F_{ij} B_j + E_i$$

exist and are finite (supposing of course the matrix of the system is non-singular and all reflectivities less than one. Additionally we suppose here all areas and reflectivities are greater than zero). Consider now $B_{max} = \max_i B_i$. Then we have

$$\begin{aligned} B_i &\leq R_i \sum_{j=1}^{n_p} F_{ij} B_{max} + E_i = R_i B_{max} \sum_{j=1}^{n_p} F_{ij} + E_i \\ &= R_i B_{max} + E_i \end{aligned}$$

Suppose now this maximum value corresponds to patch k .

Then $B_k \leq R_k B_k + E_k$ and so $B_k(1 - R_k) \leq E_k$ and as $R_k < 1$, dividing by $(1 - R_k)$ we obtain

$$B_{\max} = B_k \leq \frac{E_k}{1 - R_k} \leq \max_s \left(\frac{E_s}{1 - R_s} \right)$$

where s indexes the sources. We obtain as an upper bound for B_i

$$B_i \leq R_i B + E_i$$

calling $B = \max_s \left(\frac{E_s}{1 - R_s} \right)$. And if b_i is the reflected radiosity, we have

$$b_i = B_i - E_i \leq R_i B \leq R_{\max} B$$

This gives a bound for the radiosity of any patch due to the incoming power. Now, this bound is *independent* of how we discretize the scene in patches, supposing the sources are all diffuse. This is due to the fact that the value $\max_s \left(\frac{E_s}{1 - R_s} \right)$ is always the same in whatever subdivision of the scene we could imagine.

Suppose now that the only power we have is unit power on patch i , which means an emittance of $\frac{1}{A_i}$. Then

$$\max_s \left(\frac{E_s}{1 - R_s} \right) = \frac{1}{A_i(1 - R_i)}$$

and calling this incoming power on patch i , ξ_i , a bound for it is then given by

$$\xi_i = \frac{A_i b_i}{R_i} \leq \frac{1}{1 - R_i}$$

Note that ξ_i can be alternatively considered as the irradiance or incident radiosity on patch i due to unit emittance on the same patch i .

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