Random Walk Radiosity with generalized absorption probabilities

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Abstract

In this paper we study random walk estimators for radiosity with generalized absorption probabilities. That is, a path will either die or survive on a patch according to an arbitrary probability. The estimators studied so far, the infinite path length estimator and finite path length one, can be considered as particular cases. Practical applications of the random walks with generalized probabilities are given. A necessary and sufficient condition for the existence of the variance is given, together with an heuristics to be used in practical cases. The optimal probabilities are also found for the case when we are interested in the whole scene, and are equal to the reflectivities.

Keywords: Radiosity, Monte Carlo, Random Walk, Variance.

1. Introduction

Discrete or continuous random walk estimators have been widely used in radiosity. Gathering random walk proceeds sending paths from the patches of interest to *gather* energy when a source is hit. Path-tracing [6], and even distributed ray-tracing [3, 21] can be considered as the limiting case of gathering random walk for the non-discrete case (without the shadow ray). Shooting random walk *shoots* paths carrying energy from the sources, to update the visited patches [9], [2]. The techniques in [17, 5] can be seen as a breadth-first approach to a shooting random walk estimator, which in turn would be the depth-first approach. Bidirectional ray-tracing [20, 7] is a mixture of non-discrete shooting and gathering. The random walk proceeds according to the discrete Form Factor probability transitions. The survival (or not absorption) probability on a patch has been usually considered equal to its reflectivity. An exception to this survival probability is found in [8], where the received importance was considered instead of the reflectivity. In [10] we find also a short discussion under the term of *survival biasing*. Also, infinite path length estimators can be considered the ones where the survival probability is equal to one. We will study in this paper shooting and gathering estimators resulting of considering any survival (or not absorption) probability. We will keep the Form Factors as transition probabilities. In this way the finite path length estimators [12] and infinite ones [14] studied so far will be considered as particular cases of this generalized one.
The organization of this paper is as follows: In section 2 we present our previous work on random walk. In section 3 we study the gathering estimator with generalized absorption probabilities. A necessary and sufficient conditions for the existence of the variances is given, together with a heuristics to be used in practical situations. We give also an example of when to use this estimator. Next, in section 4, the shooting estimator is studied. The optimal survive probabilities for the case when we are interested in the whole scene are given. The resulting estimator happens to be the one with survival probability equal to the reflectivity. Finally, in section 5 we present our conclusions and future research.

2. Previous Work

In [12] we studied the three estimators defined in [18], together with their gathering dual ones. In [14] we characterized the infinite path length estimators, and in [16] we proved that the best finite path length estimator was better than the *biased* infinite path length one. Finally, in [15] we obtained the variances for the previous shooting estimators for any general source selection probability, and also proved that the results obtained so far were extensible to the pure particle tracing case, that is, when we keep the impinging point on a patch as next exiting point. The obtained results are summarized in table 1.

A feature common to all the studied estimators is that we use as transition probabilities the Form Factors, and as survive (or not absorption) probability the reflectivity of the patch (except of course for the infinite path length, where the survival probability is always 1). Here we will relax this second assumption, that is, we will consider generalized absorption probabilities. The two already studied cases, when

Table 1. Different Random Walk estimators. The meaning of the different quantities is in table 2.

Shooting	**Patch scored**	**Variance**
$\frac{\Phi_T}{1-R_i}$	last	$\sum_s \Phi_s \frac{R_i b_{is}}{A_i p_s (1-R_i)} - b_i^2$
$\frac{\Phi_T}{R_i}$	all but last	$\sum_s \Phi_s \frac{R_i b_{is}}{A_i p_s}(\frac{1}{R_i} + 2\xi_i) - b_i^2$
Φ_T	all	$\sum_s \Phi_s \frac{R_i(1+2R_i\xi_i)b_{is}}{A_i p_s} - b_i^2$
$infinite$	all	$\sum_s \Phi_s \frac{(1+2R_i\xi_i)\beta_{is}}{A_i p_s} - b_i^2$
Gathering	**Patch scored**	**Variance**
$\frac{E_s}{1-R_s}$	last	$\frac{R_i}{p_i}\sum_s \frac{E_s}{(1-R_s)} b_{is} - {b_i}^2$
$\frac{E_s}{R_s}$	all but last	$\frac{R_i}{p_i}\sum_s \frac{E_s+2b_s}{R_s} b_{is} - {b_i}^2$
E_s	all	$\frac{R_i}{p_i}\sum_s (E_s + 2b_s) b_{is} - {b_i}^2$
$infinite$	all	$\frac{1}{p_i}\sum_s (E_s + 2b_s)\beta_{is} - {b_i}^2$

Table 2. Meaning of the different quantities appearing in table 1. The suffix i means for patch i, suffix s indexes the sources.

E_i	Emissivity
b_i	Reflected radiosity = $B_i - E_i$
β_i	idem with each reflectivity substituted by its square
Φ_s	Emitted power
A_i	Area
R_i	Reflectivity
ξ_i	Received power (or radiosity) due to self-emitted unit power (or emittance)
b_{is}	Reflected radiosity on i due to source s
β_{is}	idem with each reflectivity substituted by its square
p_i	Probability for a path to begin at i

survival probability is equal to reflectivity and the infinite case, can then be seen as particular cases. We will still consider the Form Factors as transition probabilities. The usefulness of the estimators obtained can be seen when one considers survival probability proportional to importance (or better to received importance). This has been used in [8]. We will also restrict ourselves to the estimators that score on all patches. The extension of the results to the other ones, which score only on surviving patches or on the last patch, is straigthforward. Next we will proceed to study the gathering case.

3. A gathering estimator with generalized absorption probabilities

We will consider here the discrete random walk, that is, the one which proceeds according to patch-to-patch Form Factors. However, the formulae and results obtained are also valid for the point-to-point Form Factors, as shown in [15].

Let us first consider what the expected value of any unbiased Monte Carlo estimator should be for the radiosity of a patch. Let us suppose that the emittance of source s is E_s, b_i is the reflected radiosity, or radiosity of patch i due to the received power (that is, $b_i = B_i - E_i$, and so for a non-emitter patch, it equals the total radiosity), F_{kl} denotes the Form Factor from patch k to patch l, and R_k denotes the reflectance of patch k. Then we have, by developing the Radiosity system in Neumann series (dropping the zero order term):

$$\begin{aligned} b_i &= R_i \sum_s E_s F_{is} + R_i \sum_h \sum_s E_s F_{ih} R_h F_{hs} \\ &\quad + R_i \sum_h \sum_j \sum_s E_s F_{ih} R_h F_{hj} R_j F_{js} + \cdots \end{aligned}$$

This can be expressed as:

$$b_i = b_i^{(1)} + b_i^{(2)} + b_i^{(3)} + \cdots$$

where
$b_i^{(1)} = R_i \sum_s E_s F_{is}$, $b_i^{(2)} = R_i \sum_s \sum_h E_s F_{ih} R_h F_{hs}$, $b_i^{(3)} = R_i \sum_s \sum_h \sum_j E_s F_{ih} R_h F_{hj} R_j F_{js}$ and so on. That is, $b_i^{(1)}$ represents the radiosity due to direct illumination, $b_i^{(2)}$ represents the radiosity after one bounce, and so on. It is also useful to define the following quantities:

$$b_{is} = b_{is}^{(1)} + b_{is}^{(2)} + \cdots$$

$b_{is}^{(1)}$ represents the radiosity due to direct illumination from source s, $b_{is}^{(2)}$ represents the radiosity after one bounce from source s, and so on. It is clear that:

$$b_i = \sum_s b_{is}$$

Let us now consider the following simulation. A path γ starts from patch i with probability p_i (this probability can be considered as the initial or emitted importance of the patch), and from here on it evolves according to the transition probabilities given by the Form Factors. For instance,

from i it will go to patch j with probability F_{ij}. It will then be absorbed in patch j with probability $1-\theta_j$, and survive with probability θ_j. Let us define now the random variables $\widehat{b}_i^{(1)}, \widehat{b}_i^{(2)}, \widehat{b}_i^{(3)}, \ldots$ in the following way:
All of those random variables are initially null. If the path γ happens to arrive at source s at length l, and if $i, h_1, h_2, \ldots, h_{l-1}, s$ is the trajectory the path has followed, then the value of $\widehat{b}_i^{(l)}$ is set to $R_i \frac{R_{h_1}}{\theta_{h_1}} \frac{R_{h_2}}{\theta_{h_2}} \cdots \frac{R_{h_{l-1}}}{\theta_{h_{l-1}}} \frac{E_s}{p_i}$. Let us define also a new random variable $\widehat{b}_i$ as:

$$\widehat{b}_i = \widehat{b}_i^{(1)} + \widehat{b}_i^{(2)} + \widehat{b}_i^{(3)} + \cdots$$

Now let us find the expected value of those random variables. Applying the definition of expected value, and remembering that the probability of selecting patch i is p_i, the probability of landing on source s just after leaving patch i is F_{is}, we have

$$E(\widehat{b}_i^{(1)}) = \sum_s R_i \frac{E_s}{p_i} \times p_i F_{is} = b_i^{(1)}$$

Now, to go from patch i to a source s in a two length path we can pass through any patch h (after surviving on it with probability θ_h, so we have

$$\begin{aligned} E(\widehat{b}_i^{(2)}) &= \sum_h \sum_s R_i \frac{R_h}{\theta_h} \frac{E_s}{p_i} \times p_i F_{ih} \theta_h F_{hs} \\ &= b_i^{(2)} \end{aligned}$$

and so on. Then, we have

$$\begin{aligned} E(\widehat{b}_i) &= E(\widehat{b}_i^{(1)} + \widehat{b}_i^{(2)} + \cdots) = E(\widehat{b}_i^{(1)}) + E(\widehat{b}_i^{(2)}) + \cdots \\ &= b_i^{(1)} + b_i^{(2)} + \cdots = b_i \end{aligned}$$

So it is clear that the random variable $\widehat{b}_i^{(l)}$ is a centered estimator for the radiosity due to the power arrived on patch i after l bounces, and the sum of all this family of estimators gives a new centered estimator $\widehat{b}_i$ which corresponds to the total radiosity of patch i due to the power arrived after any number of bounces. Our aim now is to obtain the variance for this estimator. We will use here a similar approach to the one in [12] and [14]. We can decompose $Var(\widehat{b}_i)$ in the following way

$$\begin{aligned} Var(\widehat{b}_i) &= Var(\widehat{b}_i^{(1)} + \widehat{b}_i^{(2)} + \cdots) \\ &= E(\left(\widehat{b}_i^{(1)} + \widehat{b}_i^{(2)} + \cdots\right)^2) - \left(E(\widehat{b}_i)\right)^2 \\ &= E(\widehat{b}_i^{(1)2}) + E(\widehat{b}_i^{(2)2}) + \cdots \\ &+ 2 \sum_{1 \le n < m} E(\widehat{b}_i^{(n)} \widehat{b}_i^{(m)}) - {b_i}^2 \end{aligned} \quad (1)$$

The terms of the form $E(\widehat{b}_i^{(n)} \widehat{b}_i^{(m)})$ are not null, because if a path arrives at length n on source s it can also arrive later at source s' at length m. Next we find them:

$$\begin{aligned} & E(\widehat{b}_i^{(n)} \widehat{b}_i^{(m)}) \\ = & \sum_s \sum_{s'} \sum_{h_1} \cdots \sum_{h_{n-1}} \sum_{h_{n+1}} \cdots \sum_{h_{m-1}} R_i \frac{R_{h_1}}{\theta_{h_1}} \cdots \frac{R_{h_{n-1}}}{\theta_{h_{n-1}}} \frac{E_s}{p_i} \cdot \\ & R_i \frac{R_{h_1}}{\theta_{h_1}} \cdots \frac{R_{h_{n-1}}}{\theta_{h_{n-1}}} \frac{R_s}{\theta_s} \frac{R_{h_{n+1}}}{\theta_{h_{n+1}}} \cdots \frac{R_{h_{m-1}}}{\theta_{h_{m-1}}} \frac{E_{s'}}{p_i} \cdot \\ & p_i F_{ih_1} \theta_{h_1} \cdots F_{h_{n-1}s} \theta_s F_{sh_{n+1}} \theta_{h_{n+1}} \cdots F_{h_{m-1}s'} \\ = & \theta_i \sum_s \sum_{h_1} \cdots \sum_{h_{n-1}} \frac{R_i^2}{\theta_i} \frac{R_{h_1}^2}{\theta_{h_1}} \cdots \frac{R_{h_{n-1}}^2}{\theta_{h_{n-1}}} \frac{E_s}{p_i} \cdot \\ & p_i F_{sh_1} \cdots F_{h_{n-1}s} \cdot \\ & \sum_{s'} \sum_{h_{n+1}} \cdots \sum_{h_{m-1}} R_s F_{sh_{n+1}} R_{h_{n+1}} \cdots R_{h_{m-1}} F_{h_{m-1}s'} \frac{E_{s'}}{p_i} \\ = & \frac{\theta_i}{p_i} \sum_s \epsilon_{is}^{(n)} \sum_{s'} b_{ss'}^{(m-n)} = \frac{\theta_i}{p_i} \sum_s \epsilon_{is}^{(n)} b_s^{(m-n)} \end{aligned}$$

where $\epsilon_{is}^{(n)}$ is the radiosity due to the incoming energy after n bounces in the same environment having changed all the reflectivities by their square divided by θ_i, that is, R_i is substituted by $\frac{R_i^2}{\theta_i}$. Then

$$\begin{aligned} \sum_{1 \le n} \left(\sum_{n<m} E(\widehat{b}_i^{(n)} \widehat{b}_i^{(m)}) \right) &= \frac{\theta_i}{p_i} \sum_s \sum_{1 \le n} \epsilon_{is}^{(n)} \sum_{n<m} b_s^{(m-n)} \\ &= \frac{\theta_i}{p_i} \sum_s \epsilon_{is} b_s \end{aligned}$$

and also

$$\begin{aligned} E(\widehat{b}_i^{(1)2}) &= \sum_s \left(R_i \frac{E_s}{p_i} \right)^2 \times p_i F_{is} \\ &= \sum_s \frac{\theta_i E_s}{p_i} \frac{R_i^2}{\theta_i} F_{is} E_s \\ &= \frac{\theta_i}{p_i} \sum_s E_s \epsilon_{is}^{(1)} \end{aligned}$$

$$\begin{aligned} E(\widehat{b}_i^{(2)2}) &= \sum_h \sum_s \left(R_i \frac{R_h}{\theta_h} \frac{E_s}{p_i} \right)^2 \times p_i F_{ih} \theta_h F_{hs} \\ &= \frac{\theta_i}{p_i} \sum_s E_s \epsilon_{is}^{(2)} \end{aligned}$$

and so on. Then we obtain

$$\begin{aligned} Var(\widehat{b}_i) &= \frac{\theta_i}{p_i} \sum_s E_s (\epsilon_{is}^{(1)} + \epsilon_{is}^{(2)} + \cdots) \\ &+ 2 \frac{\theta_i}{p_i} \sum_s \epsilon_{is} b_s - {b_i}^2 \\ &= \frac{\theta_i}{p_i} \sum_s (E_s + 2 b_s) \epsilon_{is} - {b_i}^2 \end{aligned}$$

where ϵ_{is} is the radiosity due to the incoming energy in the same environment having changed all the reflectivities by their square divided by the survival (or not absorption) probability. For the radiosity our estimator is simply $\widehat{b}_i + E_i$, and as E_i is a constant we have

$$Var(\widehat{b}_i + E_i) = \frac{\theta_i}{p_i} \sum_s (E_s + 2b_s)\epsilon_{is} - {b_i}^2 \qquad (2)$$

3.1. Existence of the variance

We have that ϵ_i are the reflected radiosities solution of the same system as the original one, having substituted each reflectivity R_i by $\frac{R_i^2}{\theta_i}$. But for these quantities to exist (and be positive !) we must have as necessary and sufficient condition that the spectral radius be less than 1 [19], that is

$$\rho(\frac{R_i^2}{\theta_i} F_{ij}) < 1 \qquad (3)$$

This condition is equivalent to the one given in [4] for the existence of variance in a random walk, that is

$$lim_{k\to\infty}(norm(U^k))^{(\frac{1}{k})} < 1$$

where $U_{ij} = \frac{K_{ij}^2}{p_{ij}}$, K_{ij} is the kernel and p_{ij} are the transition probabilities (which include the survival ones). In our case $U_{ij} = \frac{R_i^2 F_{ij}^2}{\theta_i F_{ij}} = \frac{R_i^2 F_{ij}}{\theta_i}$.

The condition (3) is satisfied if for all i $\frac{R_i^2}{\theta_i} < 1$, or $\theta_i > R_i^2$. To fulfill this condition, if we are initially given survival probabilities θ_i, we just substitute them by max $(\theta_i, R_i^2 + \delta)$, where δ can be taken as small as we want (and of course, $R_i^2 + \delta$ must be less than 1).

In fact, what does the constraint $\frac{R_i^2}{\theta_i} < 1$ is to ensure the convergence of the infinite sums in formula (1). We are not claiming that without this condition there can not be convergence. What we are claiming is that, to be safe, we should take care of this condition. And of course we can show that if for all patches this condition is violated, there is no convergence. Next we prove that $\lim_{n\to\infty} E(\widehat{b}_i^{(n)2}) \neq 0$, and thus the sum can not converge.

$$\begin{aligned} E(\widehat{b}_i^{(n)2}) &= \sum_{h_1} \cdots \sum_{h_{n-1}} \sum_s \left(R_i \frac{R_{h_1}}{\theta_{h_1}} \cdots \frac{R_{h_{n-1}}}{\theta_{h_{n-1}}} \frac{E_s}{p_i} \right)^2 \\ & \quad p_i F_{i_{h_1}} \theta_{h_1} \ldots F_{i_{h_{n-1}}} \theta_{h_{n-1}} F_{hs} \\ &= \sum_{h_1} \cdots \sum_{h_{n-1}} \sum_s R_i^2 \frac{R_{h_1}^2}{\theta_{h_1}} \cdots \frac{R_{h_{n-1}}^2}{\theta_{h_{n-1}}} \frac{E_s^2}{p_i} \\ & \quad F_{i_{h_1}} \ldots F_{i_{h_{n-1}}} F_{hs} \\ &\geq \frac{R_i^2}{p_i} \sum_{h_1} \cdots \sum_{h_{n-1}} \sum_s F_{i_{h_1}} \ldots F_{i_{h_{n-1}}} F_{hs} E_s^2 \end{aligned} \qquad (4)$$

because we have supposed for all i $\frac{R_i^2}{\theta_i} \geq 1$. But we have

$$\sum_{h_1} \cdots \sum_{h_{n-1}} F_{i_{h_1}} \ldots F_{i_{h_{n-1}}} F_{hs} = (F^n)_{is}$$

and in [13] we prove that, whenever the Form Factor matrix is irreducible and aperiodic

$$\lim_{n\to\infty} (F^n)_{is} = \frac{A_s}{A_T} \qquad (5)$$

where A_T is the total area in the scene. We do not consider geometrically meaningful for a Form Factor matrix to be periodic. If there are closed rooms, it is reducible with one submatrix for each room. In this case, we consider in turn each room and we have an irreducible matrix.

But now from (4) and (5) we obtain:

$$\begin{aligned} \lim_{n\to\infty} E(\widehat{b}_i^{(n)2}) &\geq \sum_s (\lim_{n\to\infty} (F^n)_{is})\, E_s^2 \\ &= \sum_s \frac{A_s}{A_T} E_s^2 > 0 \end{aligned} \qquad (6)$$

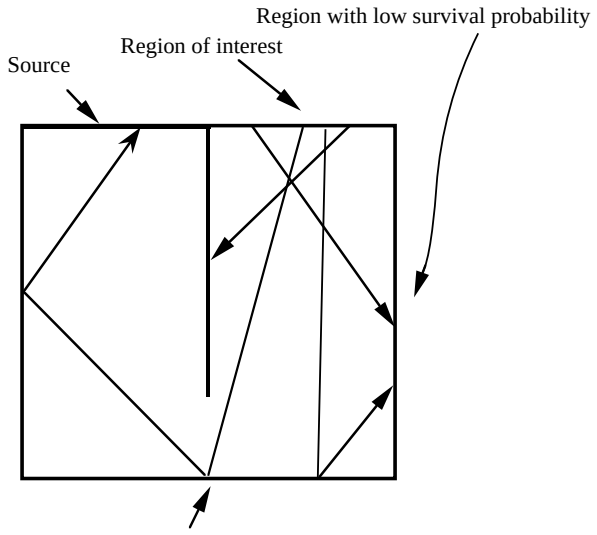

Figure 1. Paths traced from the region of interest have a higher survival probability where the reflected radiosity is higher.

As the condition $\theta_i > R_i^2$ for the existence of the variance is too restrictive, the question arises whether a weaker condition can be given. We have run simulations and found that the spectral radius (3) is approximated by, and in most cases, is less than, the average value

$$(\frac{R^2}{\theta})_{ave} = \frac{1}{A_T} \sum A_i \frac{R_i^2}{\theta_i} \qquad (7)$$

This finding is similar to the results in [1]. It is based on the following:
If we consider the series $\rho(\frac{R_i^2}{\theta_i}(F^n)_{ij})$, its limit is $\rho(\frac{R_i^2}{\theta_i}\frac{A_i}{A_T})$, but it is very easy to check that

$$\rho(\frac{R_i^2}{\theta_i}\frac{A_i}{A_T}) = \frac{1}{A_T}\sum_i A_i\frac{R_i^2}{\theta_i} = (\frac{R^2}{\theta})_{ave}$$

Thus what we do is to approximate the first term in the series, $\rho(\frac{R_i^2}{\theta_i}F_{ij})$, by the limit of the series.
This approximation will work as far as we do not have almost isolated rooms with very different reflectivities (we can not have completely isolated rooms, as we only consider here an irreducible Form Factor matrix). Excluding those cases, a heuristics such as keeping this average value less than 0.8 or maybe 0.9 should be safe.

3.2. Particular cases

When $\theta_i = 1$ we have the infinite path length estimator, $\frac{R_i^2}{\theta_i} = R_i^2$ and ϵ_{is} becomes β_{is}, and we obtain again the formula in section 2. On the other hand, when $\theta_i = R_i$ we have the E_s estimator, $\frac{R_i^2}{\theta_i} = R_i$ and ϵ_{is} becomes b_{is}, obtaining again the formula for the E_s estimator given in section 2.

3.3. An example

Suppose we have obtained a coarse solution for the radiosities. This solution could be used to drive the random walk taking $\theta_i \propto B_i - E_i$. This will assure that the paths will survive in patches with high received radiosity. This is useful in a scene alike the one in Figure 1. This case can be considered the dual of the one given in [8].

4. The shooting estimator

Let us now consider the shooting estimator with generalized absorption probabilities. Consider the following simulation. A path γ starts from source s with probability p_s, and from here on it evolves according to the transition probabilities given by the Form Factors. For instance, from s it will go to patch j with probability F_{sj}. On each patch i hit, a survival-absorption test is done according to the probabilities $\{\theta_i, 1-\theta_i\}$. If the path γ happens to arrive to the patch i at length l, then the radiosity of this patch is updated with the quantity $\frac{R_i}{p_s}\frac{R_{h_1}}{\theta_{h_1}}\frac{R_{h_2}}{\theta_{h_2}}\ldots\frac{R_{h_{l-1}}}{\theta_{h_{l-1}}}\Phi_s$.
Now, the variance can be found either using the same approach as in section 3, or taking into account that the shooting estimator can be defined as the dual of the gathering one [15]. The variance is found to be:

$$Var(\widehat{b}_i + E_i) = \theta_i\sum_s \Phi_s\frac{(1+2R_i\xi_i)\epsilon_{is}}{A_i p_s} - b_i^2 \quad (8)$$

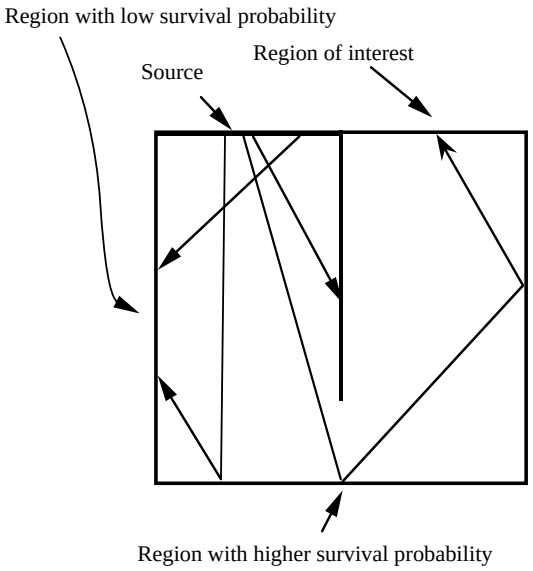

Figure 2. Paths traced from the source have a higher survival probability where the received importance towards the region of interest is higher.

Here is also valid the same argumentation as in previous section about the existence of the variance.

4.1. Particular cases

When $\theta_i = 1$ we have the infinite path length estimator, $\frac{R_i^2}{\theta_i} = R_i^2$ and ϵ_{is} becomes β_{is}, and we obtain again the formula in section 2. On the other hand, when $\theta_i = R_i$ we have the Φ_T estimator, $\frac{R_i^2}{\theta_i} = R_i$ and ϵ_{is} becomes b_{is}, obtaining again the formula for the Φ_T estimator given in section 2.
Formula (8), when taking source selection probabilities proportional to the power of the sources $p_s = \frac{\Phi_s}{\Phi_T}$ whith Φ_T total power, converts into

$$Var(\widehat{b}_i + E_i) = \theta_i\Phi_T\frac{(1+2R_i\xi_i)\epsilon_i}{A_i} - b_i^2 \quad (9)$$

where $\epsilon_i = \sum_s \epsilon_{is}$ is the reflected radiosity due to all sources changing R_i by $\frac{R_i^2}{\theta_i}$.

4.2. An example

If we make the survival probability proportional to received importance, that is, $\theta_i \propto I_i - V_i$, where the V_i is the initial and I_i the total importance [8], we will assure that the paths will survive in patches which are important to the selected ones. This could be used to drive a random walk in a scene alike the one in Figure 2, the dual of Figure 1. This case was the one considered in [8], although there a breadth-first strategy was used, instead of the considered here, which is depth-first.

4.3. Optimal survival probabilities

Suppose we are interested in all patches, not just in a single region. We want to find out the optimal survival probabilities θ_i in the sense to maximize the efficiency. This can be defined as the inverse of the product of the Variance times the cost [11], for a single patch, or the average weighted variances (expected value of the Mean Square Error) times the cost, for the whole scene. This means, taking as average cost $\frac{1}{1-\theta_{ave}}$, minimizing the quantity

$$E(MSE) \times \frac{1}{1-\theta_{ave}} \tag{10}$$

Now, using the definition of the Mean Square, formula (9) for the variance (we consider a reasonable hipothesis $p_s = \frac{\Phi_s}{\Phi_T}$ when interested on the whole scene) and the approximation

$$\epsilon_i \approx \frac{\frac{R_i^2}{\theta_i}\Phi_T}{A_T(1-(\frac{R^2}{\theta})_{ave})}$$

we obtain, following the same approach as in [16]:

$$E(MSE) \approx \frac{\Phi_T^2 R_{ave}^2}{A_T A_{ave}(1-(\frac{R^2}{\theta})_{ave})}$$

which substituted in (10) (after approximating $(\frac{R^2}{\theta})_{ave}$ by $\frac{R_{ave}^2}{\theta_{ave}}$) gives as quantity to minimize:

$$\frac{\Phi_T^2 R_{ave}^2}{A_T A_{ave}(1-(\frac{R_{ave}^2}{\theta_{ave}})}\frac{1}{1-\theta_{ave}} \tag{11}$$

The behaviour of this quantity escalated for $\frac{\Phi_T^2 R_{ave}^2}{A_T A_{ave}} = 1$ is shown in figure 3, taking the values for R_{ave} of 0.3, 0.5 and 0.8, respectively.

The analytical solution is $\theta_{ave} = R_{ave}$. This will obviously happen when for all i $\theta_i = R_i$. Thus we can state the result:

Between all the unbiased shooting random walk estimators with generalised absorption probabilities and transition probabilities the Form Factors, the most efficient for

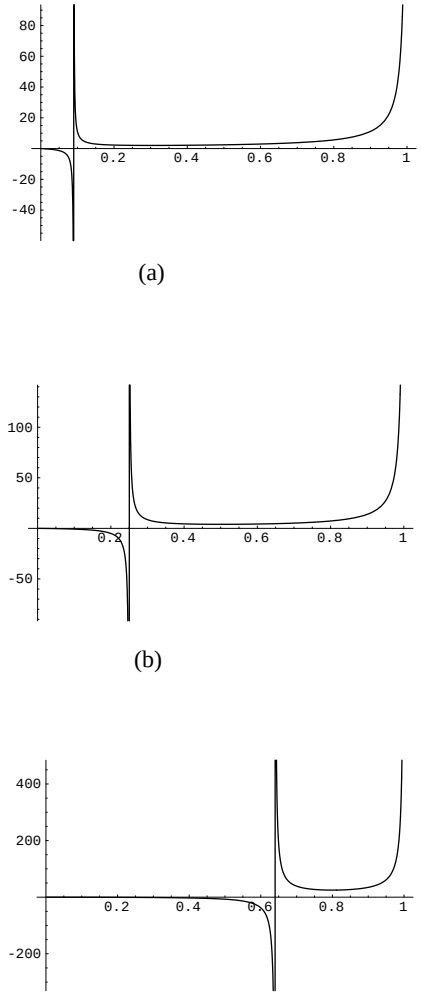

Figure 3. Behaviour of the inverse of efficiency against θ_{ave} for $R_{ave} = 0.3$, 0.5 and 0.8, respectively. The vertical asymptote corresponds to R_{ave}^2

calculating all radiosities is the one with survival probability equal to the reflectivity.

Remember that the infinite path length estimator is not considered here (the cost would be infinite), but from [16] we know that biasing it we obtain a much worse estimator than Φ_T. That means that Φ_T is the best of all shooting random walk estimators with transition probabilities the Form Factors, biased or not. And as from [12] we know that shooting estimators are much better when dealing with the whole scene than the gathering ones, we can extend this result to all random walk estimators studied till now. It must be remembered here that the Φ_T estimator in its breadth-first approach was the one used by Shirley [17] and Feda and Purgathofer [5].

5. Conclusions and future research

We have here generalized the results of [12] and [14] to the case of a generalized absorption probability, obtaining closed formulae for the variances of the estimators studied. Those are presented in table 3. A necessary and sufficient condition for the existence of the variance is also given, together with a heuristics to be used in practical cases. The usefulness of such estimators has also been shown. It has been proved that the best shooting estimator, when we are interested in the whole scene, comes from using the reflectivity as the survival probability. Future work will be directed towards obtaining closed formulae, whenever possible, for generalized transition probabilities.

Table 3. Variances for Random Walk estimators with generalized absorption probabilities, θ_i. ϵ_{is} is the reflected radiosity in patch i due to source s substituting each reflectivity by $\frac{R_i}{\theta_i}$.

$shooting$	$\theta_i \sum_s \Phi_s \frac{(1+2R_i\xi_i)\epsilon_{is}}{A_i p_s} - b_i^2$
$gathering$	$\frac{\theta_i}{p_i} \sum_s (E_s + 2b_s)\epsilon_{is} - {b_i}^2$

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