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An LMI Approach to H_∞ Synchronization of Second-Order Neutral Master-Slave Systems

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Abstract— **The H_∞ synchronization problem of the master and slave structure of a second-order neutral master-slave systems with time-varying delays is presented in this paper. Delay-dependent sufficient conditions for the design of a delayed output-feedback control are given by Lyapunov-Krasovskii method in terms of a linear matrix inequality (LMI). A controller, which guarantees H_∞ synchronization of the master and slave structure using some free weighting matrices, is then developed. A numerical example has been given to show the effectiveness of the method.**

I. INTRODUCTION

In the last few years, synchronization in chaotic dynamical systems has received a great deal of interest among scientists from various fields [1, 2]. The results of chaos synchronization are utilized in biology, secret communication and cryptography, nonlinear oscillation synchronization and some other nonlinear fields. The first idea of synchronizing two identical chaotic systems with different initial conditions was introduced by Pecora and Carroll [3], and the method was realized in electronic circuits. The methods for synchronization of the chaotic systems have been widely studied in recent years, and many different methods have been applied theoretically and experimentally to synchronize chaotic systems, such as feedback control [4-8], adaptive control [9, 10], backstepping [11] and sliding mode control [12].

One of the most attractive dynamical systems is the second-order systems which capture the dynamic behaviour of many natural phenomena, and have found applications in many fields, such as vibration and structural analysis, spacecraft control, electrical networks, robotics control and, hence, have attracted much attention (see, [13-16]). It has been proved that in special situations a second-order system may show chaotic dynamics, for instance, in [17], a second-order linear plant containing a relay with hysteresis type nonlinearity shows the chaotic nature of its dynamical behavior. Moreover, complex dynamical behavior of second-order linear plants controlled with conventional controllers is investigated in [18, 19]. On the other hand, in view of the time-delay phenomenon, which is frequently encountered in practical situations, this delay may induce complex behaviors for the systems concerned (see [20, 21]). Up to now, to the best of the authors' knowledge, no results about the synchronization of second-order master-slave systems with time-varying delays using delayed output-feedback control are available in the literature, which remains to be important and challenging.

In this paper, we make an attempt to develop an efficient approach for H_∞ synchronization problem of second-order neutral master-slave systems with time-varying state delays. The main merit of the proposed method lies in the fact that it provides a convex problem via introduction of additional decision variables such that the control gains can be found from the LMI formulations without reformulating the system equations into a standard form of a first-order neutral system. By using a Lyapunov-Krasovskii method and some free weighting matrices, new sufficient conditions are established in terms of a delay-dependent LMI for the existence of desired delayed output-feedback control such that the resulting closed-loop system is asymptotically stable and satisfies a prescribed H_∞ performance. A significant advantage of our result is that the desired control is designed directly instead of coupling the model to a first-order neutral system and then designing the control law in a higher dimensional space. Therefore, our result can be implemented in a numerically stable and efficient way for high-dimensional second-order systems. Furthermore, retaining the model in matrix second-order form has many advantages such as preserving physical insight of the original problem, preserving system matrix sparsity and structure, preserving uncertainty structure and entailing easier implementation. Finally, the simulation results are given to illustrate the usefulness of our results.

II. PROBLEM DESCRIPTION

Consider a model of second-order neutral master-slave systems in the form of

$$\begin{cases} M\,\ddot{x}_m(t) + M_1\,\ddot{x}_m(t-d(t)) + A\,\dot{x}_m(t) + A_1\,\dot{x}_m(t-r(t)) + B\,x_m(t) \\ \qquad + B_1\,x_m(t-r(t)) + N_1 f(x_m(t)) + N_2 g(x_m(t-r(t))) = 0, \\ x_m(t) = \phi(t), \qquad t \in [-\max\{d_M, r_M\},\ 0], \\ \dot{x}_m(t) = \dot{\phi}(t), \qquad t \in [-\max\{d_M, r_M\},\ 0], \\ z_m(t) = C_1\,x_m(t) + C_2\,x_m(t-r(t)), \\ y_m(t) = C_3\,x_m(t), \end{cases} \tag{1}$$

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$$\begin{cases} M\ddot{x}_s(t)+M_1\ddot{x}_s(t-d(t))+A\dot{x}_s(t)+A_1\dot{x}_s(t-r(t))+Bx_s(t) \\ \quad +B_1x_s(t-r(t))+N_1f(x_s(t))+N_2g(x_s(t-r(t)))=B_2u(t)+Dw(t), \\ x_s(t)=\varphi(t), \qquad t\in[-\max\{d_M,r_M\},\,0], \\ \dot{x}_s(t)=\dot{\varphi}(t), \qquad t\in[-\max\{d_M,r_M\},\,0], \\ z_s(t)=C_1x_s(t)+C_2x_s(t-r(t)), \\ y_s(t)=C_3x_s(t), \end{cases} \tag{2}$$

where $x_m(t), x_s(t)$ are the $n\times 1$ state vector of the master and slave systems, respectively; $u(t)$ is the $r\times 1$ control input; $w(t)$ is the $q\times 1$ external excitation (disturbance), $z_m(t), z_s(t)$ are the $s\times 1$ controlled output and $y_m(t), y_s(t)$ is the $l\times 1$ measured output. The time-varying vector valued initial functions $\phi(t)$ and $\varphi(t)$ are continuously differentiable functionals, and the time-varying delays $d(t)$ and $r(t)$ are functions satisfying, respectively,

$$\begin{cases} 0<d(t)\le d_M, & \dot{d}(t)\le d_D<1, \\ 0<r(t)\le r_M, & \dot{r}(t)\le r_D. \end{cases} \tag{3}$$

Assumption 1: The nonlinear functions $f, g:\Re^n\to\Re^n$ are continuous and satisfy $f(0)=g(0)=0$ and the Lipschitz condition, i.e., $\|\underline{f}(x_0-y_0)\|\le\|f(x_0)-f(y_0)\|\le\|\bar{f}(x_0-y_0)\|$ and $\|\underline{g}(x_0-y_0)\|\le\|g(x_0)-g(y_0)\|\le\|\bar{g}(x_0-y_0)\|$ for all $x_0, y_0\in\Re^n$ and $\bar{f}, \underline{f}, \bar{g}, \underline{g}$ are some known matrices.

Now, the synchronization error of the master and slave systems (1) and (2) is defined as $e(t)=x_s(t)-x_m(t)$, then the error dynamics between (1) and (2), namely synchronization error system, can be expressed by

$$\begin{cases} M\ddot{e}(t)+M_1\ddot{e}(t-d(t))+A\dot{e}(t)+A_1\dot{e}(t-r(t))+Be(t) \\ \quad +B_1e(t-r(t))+N_1\hat{f}(e(t))+N_2\hat{g}(e(t-r(t)))=B_2u(t)+Dw(t), \\ z_e(t)=C_1e(t)+C_2e(t-r(t)), \\ y_e(t)=C_3e(t), \end{cases} \tag{4}$$

where $z_e(t)=z_s(t)-z_m(t)$, $\hat{f}(e(t)):=f(x_s(t))-f(x_m(t))$ and $\hat{g}(e(t-r(t))):=g(x_s(t-r(t)))\ -g(x_m(t-r(t)))$.

The problem to be addressed in this paper is formulated as follows: given the second-order neutral master-slave systems (1) and (2) with any time-varying delays satisfying (3) and a prescribed level of disturbance attenuation $\gamma>0$, find a delayed output-feedback control $u(t)$ of the form

$$\begin{aligned} u(t) &= K_1y_e(t)+K_2\dot{y}_e(t)+K_3y_e(t-r(t))+K_4\dot{y}_e(t-r(t)) \\ &:= KC\xi(t)+K_rC\xi(t-r(t)) \end{aligned} \tag{5}$$

where $K:=[K_1\quad K_2]$, $K_r:=[K_3\quad K_4]$, $C:=diag\{C_3,C_3\}$, $\xi(t)=col\{e(t),\dot{e}(t)\}$ and the matrices $\{K_i\}_{i=1}^4$ are the control gains to be determined such that

1) the synchronization error system (4) is asymptotically stable for any time delays satisfying (3);
2) under zero initial conditions and for all non-zero $w(t)\in L_2[0,\infty]$, the H_∞ performance measure, i.e., $J_\infty=\int_0^\infty z_e^T(t)z_e(t)-\gamma^2w^T(t)w(t)\ \ dt$, satisfies $J_\infty<0$;

in this case, the systems (1) and (2) are said to be asymptotically stable with H_∞ performance measures.

III. MAIN RESULTS

In this section, sufficient conditions for the solvability of the delayed output-feedback control design problem are proposed using the Lyapunov method and an LMI approach.

Lemma 1 ([22]): For any arbitrary positive definite matrix H and a matrix W the following inequality holds:

$$-2\int_{t-r(t)}^{t}b(s)^Ta(s)\,ds\le\int_{t-r(t)}^{t}\begin{bmatrix}a(s)\\b(s)\end{bmatrix}^T\begin{bmatrix}H & HW\\ * & (HW+I)^TH^{-1}(HW+I)\end{bmatrix}\begin{bmatrix}a(s)\\b(s)\end{bmatrix}ds$$

where the symbol $*$ denotes the elements below the main diagonal of a symmetric block matrix.

Lemma 2: ([23]) For a given $\mathrm{M}\in\Re^{p\times n}$ with $rank\,(\mathrm{M})=p<n$, assume that $Z\in\Re^{n\times n}$ is a symmetric matrix, then there exists a matrix $\hat{Z}\in\Re^{p\times p}$ such that $\mathrm{M}Z=\hat{Z}\mathrm{M}$ if and only if

$$Z=V.diag\{Z_1,Z_2\}.V^T \text{ and } \hat{Z}=U\hat{\mathrm{M}}Z_1\hat{\mathrm{M}}^{-1}U^T,$$

where $Z_1\in\Re^{p\times p}$, $Z_2\in\Re^{(n-p)\times(n-p)}$ and the singular value decomposition of the matrix M is represented as $\mathrm{M}=U[\hat{\mathrm{M}}\quad 0]V^T$ with the unitary matrices $U\in\Re^{p\times p}$, $V\in\Re^{n\times n}$ and a diagonal matrix $\hat{\mathrm{M}}\in\Re^{p\times p}$ with positive diagonal elements in decreasing order.

Theorem 1: For given scalars $d_M, r_M>0$, $d_D<1$, r_D and $\gamma>0$, the second-order neutral master-slave systems (1) and (2) with any time-varying delays satisfying (3) is robustly stabilizable by (5) and satisfies the H_∞ performance measure, if there exist some matrices P_2, P_3, W, $\{F_i\}_{i=1}^2$, positive-definite matrices P_1, $\{Q_i\}_{i=1}^2$, H and positive-definite diagonal matrices $\{\Lambda_i\}_{i=1}^3$, such that the following inequality is feasible,

$$\Pi=\begin{bmatrix}\Pi_{11}+\begin{bmatrix}(C_1\hat{I})^T\\0\end{bmatrix}[C_1\hat{I}\quad 0] & \Pi_{12} & \Pi_{13}\\ * & \Pi_{22} & \Pi_{23}\\ * & * & \Pi_{33}\end{bmatrix}<0 \tag{6}$$

with $\Pi_{12}=P^T(\begin{bmatrix}0\\\hat{A}_2\end{bmatrix}-W^TH\begin{bmatrix}0\\A_1\end{bmatrix}\tilde{I})+\begin{bmatrix}F_2^T-F_1+(C_1\hat{I})^TC_2\hat{I}\\0\end{bmatrix}$,

$\Pi_{22}=-(1-r_D)Q_1-sym\{F_2\}-sym\{(\underline{g}\ \hat{I})^T\Lambda_3\underline{g}\ \hat{I}\}+\hat{I}^TC_2^TC_2\hat{I}$,

$\Pi_{13}:=\left[P^T\begin{bmatrix}0\\M_1\end{bmatrix},P^T\begin{bmatrix}0\\N_1\end{bmatrix}+\begin{bmatrix}\hat{I}^T(\bar{f}+\underline{f})^T\Lambda_1\\0\end{bmatrix},\begin{bmatrix}\hat{I}^T(\bar{g}+\underline{g})^T\Lambda_2\\0\end{bmatrix},\right.$

$\left.P^T\begin{bmatrix}0\\N_2\end{bmatrix},-P^T\begin{bmatrix}0\\D\end{bmatrix},\begin{bmatrix}F_1\\0\end{bmatrix}\right]$, $\Pi_{23}:=\left[0,0,0,\hat{I}^T(\bar{g}+\underline{g})^T\Lambda_3,0,F_2\right]$,

$$\Pi_{33} := diag\{-(1-d_D)Q_2, -sym\{\Lambda_1\}, -sym\{\Lambda_2\}, -sym\{\Lambda_3\}, -\gamma^2 I, -r_M P_1\},$$

$$\Pi_{11} := sym\{P^T \begin{bmatrix} \hat{I}^T\tilde{I} & \tilde{I}^T \\ \hat{A}_1 & M \end{bmatrix}\} - sym\{\begin{bmatrix} I \\ 0 \end{bmatrix}\hat{I}^T \bar{f}^T \Lambda_1 \underline{f}\,\hat{I}\} - sym\{\begin{bmatrix} I \\ 0 \end{bmatrix}\hat{I}^T \bar{g}^T \Lambda_2 \underline{g}\,\hat{I}\}$$
$$+ sym\{P^T W^T H \begin{bmatrix} 0 \\ A_1 \end{bmatrix}[\tilde{I} \quad 0]\} + r_M P^T (W^T H + I) H^{-1} (HW + I) P$$
$$+ diag\{Q_1 + sym\{F_1\}, Q_2\} + r_M \begin{bmatrix} \tilde{I} & 0 \\ 0 & I \end{bmatrix}^T P_1 \begin{bmatrix} \tilde{I} & 0 \\ 0 & I \end{bmatrix}$$
$$+ \frac{r_M}{1-r_D}\begin{bmatrix} 0 \\ I \end{bmatrix}\begin{bmatrix} 0 \\ A_1 \end{bmatrix}^T H \begin{bmatrix} 0 \\ A_1 \end{bmatrix}[0 \quad I]$$

where $\hat{A}_1 := B\hat{I} + (A + A_1)\tilde{I} - B_2 K C$, $\hat{A}_2 := [B_1 \quad 0] - B_2 K_r C$, the operator $sym\{A\}$ denotes $A + A^T$ and the matrices $\hat{I}$ and $\tilde{I}$ are defined, respectively, as $\hat{I} := [I \quad 0]$ and $\tilde{I} := [0 \quad I]$.

Proof: Firstly, we represent the synchronization error system (4) in an equivalent descriptor model form as

$$\begin{cases} \dot{e}(t) = \eta(t), \\ 0 = M\eta(t) + M_1\eta(t-d(t)) + \hat{A}_1\xi(t) + \hat{A}_2\xi(t-r(t)) - A_1\int_{t-r(t)}^{t}\eta(s)\,ds \\ \quad + N_1\hat{f}(e(t)) + N_2\hat{g}(e(t-r(t))) - Dw(t), \end{cases} \tag{7}$$

Define the Lyapunov-Krasovskii functional

$$V(t) = \sum_{i=1}^{5} V_i(t), \tag{8}$$

where

$$V_1(t) = \xi(t)^T P_1 \xi(t) := [\xi(t)^T \quad \eta(t)^T] T P \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix},$$

$$V_2(t) = \int_{t-r(t)}^{t} \xi(s)^T Q_1 \xi(s)\,ds, V_3(t) = \int_{t-d(t)}^{t} \eta(s)^T Q_2 \eta(s)\,ds,$$

$$V_4(t) = \int_{-r(t)}^{0}\int_{t+\theta}^{t} \dot{\xi}(s)^T P_1 \dot{\xi}(s)\,ds\,d\theta,$$

$$V_5(t) = \frac{1}{1-r_D}\int_{t-r(t)}^{t} (s-t+r(t))\eta(s)^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix}^T H \begin{bmatrix} 0 \\ A_1 \end{bmatrix}\eta(s)\,ds,$$

with $T = diag\{I, 0\}$ and $P = \begin{bmatrix} P_1 & 0 \\ P_3 & P_2 \end{bmatrix}$, where $0 < P_1 = P_1^T$.

Differentiating $V_1(t)$ along the system trajectory becomes

$$\dot{V}_1(t) = 2\xi(t)^T P_1 \dot{\xi}(t) = 2[\xi(t)^T \quad \eta(t)^T]\, P^T \begin{bmatrix} \dot{\xi}(t) \\ 0 \end{bmatrix}$$
$$= 2[\xi(t)^T \quad \eta(t)^T]\, P^T \{\begin{bmatrix} \hat{I}^T\tilde{I} & \tilde{I}^T \\ \hat{A}_1 & M \end{bmatrix}\begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ M_1 \end{bmatrix}\eta(t-d(t)) + \begin{bmatrix} 0 \\ \hat{A}_2 \end{bmatrix} \tag{9}$$
$$\times\xi(t-r(t)) + \begin{bmatrix} 0 \\ N_1 \end{bmatrix}\hat{f}(e(t)) + \begin{bmatrix} 0 \\ N_2 \end{bmatrix}\hat{g}(e(t-r(t))) - \begin{bmatrix} 0 \\ D \end{bmatrix}w(t)\} + \beta(t)$$

where $\beta(t) = -2\int_{t-r(t)}^{t} [\xi(t)^T \quad \eta(t)^T] P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix}\eta(s)\,ds$. Using Lemma 1 for $a(s) = col\{0, A_1\}\eta(s)$ and $b = P\,col\{\xi(t), \eta(t)\}$ we obtain

$$\beta(t) \le r_M[\xi(t)^T \quad \eta(t)^T] P^T (W^T H + I) H^{-1} (HW + I) P \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix}$$
$$+ 2[\xi(t)^T \quad \eta(t)^T] P^T W^T H \begin{bmatrix} 0 \\ A_1 \end{bmatrix}\tilde{I}(\xi(t) - \xi(t-r(t))) \tag{10}$$
$$+ \int_{t-r(t)}^{t} \eta(s)^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix}^T H \begin{bmatrix} 0 \\ A_1 \end{bmatrix}\eta(s)\,ds$$

Also, differentiating the second to forth Lyapunov terms in (8) give

$$\dot{V}_2(t) \le \xi(t)^T Q_1 \xi(t) - (1-r_D)\xi^T(t-r(t))Q_1\xi(t-r(t)), \tag{11}$$

$$\dot{V}_3(t) \le \eta(t)^T Q_2 \eta(t) - (1-d_D)\eta^T(t-d(t))Q_2\eta(t-d(t)), \tag{12}$$

$$\dot{V}_4(t) \le r_M\,\dot{\xi}(t)^T P_1 \dot{\xi}(t) - \int_{t-r(t)}^{t} \dot{\xi}(s)^T P_1 \dot{\xi}(s)\,ds \tag{13}$$

and the time derivative of the last term of $V(t)$ in (8) is

$$\dot{V}_5(t) \le \frac{r_M}{1-r_D}\eta(t)^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix}^T H \begin{bmatrix} 0 \\ A_1 \end{bmatrix}\eta(t) - \int_{t-r(t)}^{t} \eta(s)^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix}^T H \begin{bmatrix} 0 \\ A_1 \end{bmatrix}\eta(s)\,ds \tag{14}$$

Moreover, from the Leibniz-Newton formula, the following equation holds for any matrices $\{F_1\}_{i=1}^{2}$ with appropriate dimensions:

$$2(\xi^T(t)F_1 + \xi^T(t-r(t))F_2)(\xi(t) - \xi(t-r(t)) - \int_{t-r(t)}^{t} \dot{\xi}(s)\,ds) = 0 \tag{15}$$

On the other hand, for any positive scalars $\{\Lambda_i\}_{i=1}^{3}$ we have:

$$0 < -2(\hat{f}(e(t)) - \bar{f}\,e(t))^T \Lambda_1 (\hat{f}(e(t)) - \underline{f}\,e(t))$$
$$0 < -2(\hat{g}(e(t)) - \bar{g}\,e(t))^T \Lambda_2 (\hat{g}(e(t)) - \underline{g}\,e(t))$$
$$0 < -2(\hat{g}(e(t-r(t))) - \bar{g}\,e(t-r(t)))^T \Lambda_3 (\hat{g}(e(t-r(t))) - \underline{g}\,e(t-r(t))) \tag{16}$$

Using the obtained derivative terms (9)-(15) and adding the right-hand sides of equation (16) into, we obtain the following result for $\dot{V}(t)$,

$$\dot{V}(t) = \sum_{i=1}^{5}\dot{V}_i(t) \le 2[\xi(t)^T \quad \eta(t)^T]\, P^T \{\begin{bmatrix} \hat{I}^T\tilde{I} & \tilde{I}^T \\ \hat{A}_1 & M \end{bmatrix}\begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ M_1 \end{bmatrix}$$
$$\times\eta(t-d(t)) + \begin{bmatrix} 0 \\ \hat{A}_2 \end{bmatrix}\xi(t-r(t)) + \begin{bmatrix} 0 \\ N_1 \end{bmatrix}\hat{f}(e(t)) + \begin{bmatrix} 0 \\ N_2 \end{bmatrix}\hat{g}(e(t-r(t)))$$
$$- \begin{bmatrix} 0 \\ D \end{bmatrix}w(t)\} + r_M[\xi(t)^T \quad \eta(t)^T] P^T (W^T H + I) H^{-1} (HW + I) P \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix}$$
$$+ 2[\xi(t)^T \quad \eta(t)^T] P^T W^T H \begin{bmatrix} 0 \\ A_1 \end{bmatrix}\tilde{I}(\xi(t) - \xi(t-r(t))) + \frac{r_M}{1-r_D}\eta(t)^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix}^T$$
$$\times H \begin{bmatrix} 0 \\ A_1 \end{bmatrix}\eta(t) + \xi(t)^T(Q_1 + F_1 + F_1^T)\xi(t) - \xi^T(t-r(t))((1-r_D)Q_1 + F_2$$
$$+ F_2^T)\xi(t-r(t)) + 2\xi(t)^T(F_2^T - F_1)\xi(t-r(t)) + r_M\,\dot{\xi}(t)^T P_1 \dot{\xi}(t) + \eta(t)^T Q_2$$
$$\times\eta(t) - (1-d_D)\eta^T(t-d(t))Q_2\eta(t-d(t)) + r_M\vartheta^T(t)F P_1^{-1} F^T \vartheta(t)$$
$$- 2(\hat{f}(e(t)) - \bar{f}\,\hat{I}\xi(t))^T \Lambda_1 (\hat{f}(e(t)) - \underline{f}\,\hat{I}\xi(t)) - 2(\hat{g}(e(t)) - \bar{g}\,\hat{I}\xi(t))^T \Lambda_2$$
$$\times(\hat{g}(e(t)) - \underline{g}\,\hat{I}\xi(t)) - 2(\hat{g}(e(t-r(t))) - \bar{g}\,\hat{I}\xi(t-r(t)))^T \Lambda_3 (\hat{g}(e(t-r(t)))$$
$$- \underline{g}\,\hat{I}\xi(t-r(t))) - \int_{t-r(t)}^{t} (\vartheta^T(t)F + \dot{\xi}^T(s)P_1) P_1^{-1} (\vartheta^T(t)F + \dot{\xi}^T(s)P_1)^T\,ds \tag{17}$$

where the vectors $\vartheta(t)$ and N are, respectively,

$$\vartheta(t) := col\{\xi(t), \eta(t), \xi(t-r(t)), \eta(t-d(t)), w(t)\}, \tag{18}$$

$$F := col\{F_1, 0, F_2, 0, 0\}. \tag{19}$$

Under zero initial conditions, the H_∞ performance measure can be rewritten as

$$J_\infty = \int_0^\infty [(z_s(t)-z_m(t))^T (z_s(t)-z_m(t)) - \gamma^2 w(t)^T w(t) + \dot{V}(t)]\, dt \tag{20}$$

Substituting the terms of $\dot{\xi}(t) = \begin{bmatrix} \hat{I}^T\tilde{I} & \tilde{I}^T \end{bmatrix} col\{\xi(t), \eta(t)\}$, $z_s(t)-z_m(t) = C_1\hat{I}\,\xi(t) + C_2\,\hat{I}\,\xi(t-r(t))$, and upper bound of $\dot{V}(t)$ in (17) results in (20) being less than the integrand $\vartheta(t)^T\,\Pi\,\vartheta(t)$ where the matrix Π, by Schur complement, is given in (6). Now, if $\Pi < 0$, then $J_\infty < 0$. ■

Remark 1: It is easy to see that the inequality (6) imply $\Pi_{11} < 0$. Hence by Proposition 4.2 in the reference [24], the matrix P is nonsingular. Then, according to the structure of the matrix P, the matrix $X := P^{-1}$ has the form

$$X = \begin{bmatrix} X_1 & 0 \\ X_3 & X_2 \end{bmatrix},$$

where $X_i = P_i^{-1}$ $(i = 1, 2)$ and $X_3 = -X_2 P_3 X_1$.

Remark 2: According to structure of matrix C, i.e., $C := diag\{C_3, C_3\}$, with $rank(C_3) = l < n$, Lemma 2 proposes that an equivalent condition on equation $CX_1 = \hat{X}_1 C$ is

$$X_1 = V\, .diag\{X_{11}, X_{22}\} V^T,\ \hat{X}_1 = U\hat{C}X_{11}\hat{C}^{-1}U^T,$$

where $X_{11} \in \Re^{2l\times 2l}$, $X_{22} \in \Re^{2(n-l)\times 2(n-l)}$ and $C = U[\hat{C}\ \ 0]V^T$ (the singular value decomposition of the matrix C), with $rank(C) = 2l$, $U \in \Re^{2l\times 2l}$, $V \in \Re^{2n\times 2n}$ and $\hat{C} \in \Re^{2l\times 2l}$.

Theorem 2: Consider the second-order neutral master-slave systems (1) and (2) with any time-varying delays satisfying (3). For given scalars $d_M, r_M > 0$, $d_D < 1$, r_D and $\gamma > 0$, there exits an output-feedback control in the form of (5) such that the resulting closed-loop system is robustly asymptotically stable and satisfies H_∞ performance measure in Definition 1, if there exist a scalar α, matrices $\{\hat{F}_i\}_{i=1}^2$, $\{\tilde{X}_i\}_{i=1}^2$, X_2, X_3, positive-definite matrices X_{11}, X_{22}, $\{\hat{Q}_i\}_{i=1}^2$, $\overline{H}$ and positive definite diagonal matrices $\{\overline{\Lambda}_i\}_{i=1}^3$, satisfying the LMI

$$\begin{bmatrix} \hat{\Pi}_{11} & \hat{\Pi}_{12} & \hat{\Pi}_{13} & \hat{\Pi}_{14} & \hat{\Pi}_{15} \\ * & -(1-r_D)\hat{Q}_1 - sym\{\hat{F}_2\} & \hat{\Pi}_{23} & 0 & \hat{\Pi}_{25} \\ * & * & \hat{\Pi}_{33} & 0 & \hat{\Pi}_{35} \\ * & * & * & \hat{\Pi}_{44} & \hat{\Pi}_{45} \\ * & * & * & * & \hat{\Pi}_{55} \end{bmatrix} < 0 \tag{21}$$

where

$$\hat{\Pi}_{11} := \begin{bmatrix} \begin{pmatrix} \hat{I}^T\tilde{I}X_1 + X_1\tilde{I}^T\hat{I} + \hat{F}_1 + \hat{F}_1^T \\ +\tilde{I}^T X_3 + X_3^T\tilde{I} + \hat{Q}_1 \end{pmatrix} & \begin{pmatrix} X_1(B\hat{I} + (A+(1+\alpha)A_1)\tilde{I})^T \\ -C^T\tilde{X}_1^T B_2^T + \tilde{I}^T X_2 + X_3^T M^T \end{pmatrix} \\ * & M X_2 + X_2^T M^T \end{bmatrix}$$

$$\hat{\Pi}_{13} = \left[\begin{bmatrix} 0 \\ M_1 \end{bmatrix} X_1, \begin{bmatrix} 0 \\ N_1 \end{bmatrix} \overline{\Lambda}_1^T + X^T \begin{bmatrix} \hat{I}^T(\bar{f}+\underline{f})^T \\ 0 \end{bmatrix}, X^T \begin{bmatrix} \hat{I}^T(\bar{g}+\underline{g})^T \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ N_2 \end{bmatrix} \overline{\Lambda}_3^T \right]$$

$$\hat{\Pi}_{23} = \left[0,0,0, X_1\hat{I}^T(\bar{g}+\underline{g})^T\right], \hat{\Pi}_{12} := \begin{bmatrix} \hat{F}_2^T - \hat{F}_1 \\ (B_1\hat{I} - \alpha A_1\tilde{I})X_1 - B_2\tilde{X}_2 C \end{bmatrix}$$

$$\hat{\Pi}_{33} = diag\{-(1-d_D)\hat{Q}_2, -sym\{\overline{\Lambda}_1\}, -sym\{\overline{\Lambda}_2\}, -sym\{\overline{\Lambda}_3\}\}$$

$$\hat{\Pi}_{14} = \left[-\begin{bmatrix} 0 \\ D \end{bmatrix}, \begin{bmatrix} \hat{F}_1 \\ 0 \end{bmatrix}, r_M \begin{bmatrix} X_1\tilde{I}^T\hat{I} + X_3^T\tilde{I} \\ X_2^T\tilde{I} \end{bmatrix}, \begin{bmatrix} (C_1\hat{I})^T \\ 0 \end{bmatrix}, r_M(\alpha+1)\overline{H} \right]$$

$$\hat{\Pi}_{44} = diag\{-\gamma^2 I, -r_M X_1, -r_M X_1, -I, -r_M\overline{H}\}$$

$$\hat{\Pi}_{15} = \left[\begin{bmatrix} X_3^T \\ X_2^T \end{bmatrix}, \begin{bmatrix} 0 & X_3^T A_1^T \\ 0 & X_2^T A_1^T \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \hat{I}^T\bar{f} & \hat{I}^T\underline{f} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \hat{I}^T\bar{g} & \hat{I}^T\underline{g} \end{bmatrix}, 0 \right]$$

$$\hat{\Pi}_{25} = \left[0,0,0,0, \begin{bmatrix} \hat{I}^T\bar{g} & \hat{I}^T\underline{g} \end{bmatrix}\right]$$

$$\hat{\Pi}_{55} = diag\{-\overline{Q}_2, -\frac{1-r_D}{r_M}\overline{H}, -\overline{\Lambda}_1, -\overline{\Lambda}_1, -\overline{\Lambda}_2, -\overline{\Lambda}_2, -\overline{\Lambda}_3, -\overline{\Lambda}_3\}$$

The desired control gains in (5) are given by

$$K = \tilde{X}_1\hat{X}_1^{-1},\ K_r = \tilde{X}_2\hat{X}_1^{-1} \text{ from LMI (21)}, \tag{22}$$

where the matrices X_1 and $\hat{X}_1$ follow from Remark 2.

Proof: Let $\zeta = diag\{X^T, X_1, X_1, \overline{\Lambda}_1, \overline{\Lambda}_2, \overline{\Lambda}_3, I, X_1, X_1, I, \overline{H}\}$ where $\overline{\Lambda}_i := \Lambda_i^{-1}$ and $\overline{H} = H^{-1}$. By introducing $T := HWP$ as a new decision variable (with $TX = \alpha I$), applying the Schur complement to the matrix inequality (6) in Theorem 1 and premultiplying ζ and postmultiplying ζ^T where $\overline{\Lambda}_i := \Lambda_i^{-1}$ and $\overline{H} = H^{-1}$ and using the inequalities

$$-sym\left\{\begin{bmatrix} I \\ 0 \end{bmatrix}\hat{I}^T\bar{f}\,\Lambda_1\underline{f}\,\hat{I}\right\} \le \begin{bmatrix} I \\ 0 \end{bmatrix}\hat{I}^T\bar{f}\,\Lambda_1\bar{f}\,\hat{I}\begin{bmatrix} I \\ 0 \end{bmatrix}^T + \hat{I}^T\underline{f}\,\Lambda_1\underline{f}\,\hat{I}$$

$$-sym\left\{\begin{bmatrix} I \\ 0 \end{bmatrix}\hat{I}^T\bar{g}\,\Lambda_2\underline{g}\,\hat{I}\right\} \le \begin{bmatrix} I \\ 0 \end{bmatrix}\hat{I}^T\bar{g}\,\Lambda_2\bar{g}\,\hat{I}\begin{bmatrix} I \\ 0 \end{bmatrix}^T + \hat{I}^T\underline{g}\,\Lambda_2\underline{g}\,\hat{I}$$

$$-sym\{\hat{I}^T\bar{g}\,\Lambda_3\,\underline{g}\,\hat{I}\} \le \hat{I}^T\bar{g}\,\Lambda_3\,\bar{g}\,\hat{I} + \hat{I}^T\underline{g}\,\Lambda_3\underline{g}\,\hat{I} \tag{23}$$

and considering $\tilde{X}_1 := K\hat{X}_1$, $\hat{Q}_i = X_1^T Q_i X_1$ and $\hat{F}_i = X_1^T F_i X_1$, we obtain the LMI (21). ■

IV. SIMULATION RESULTS

Consider the second-order neutral master-slave systems (1) and (2), where the system matrices are given by

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0.8 \end{bmatrix},\ M_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix},\ A = \begin{bmatrix} 1 & 0.5 \\ 0.3 & 0.5 \end{bmatrix},$$

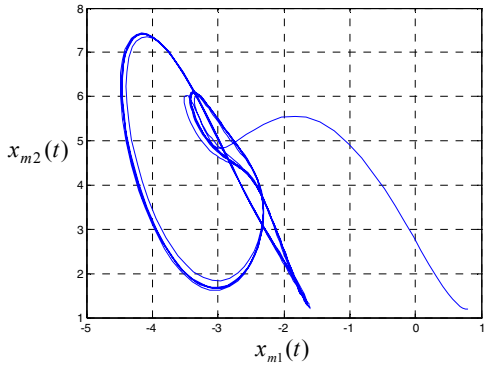
$$A_1 = \begin{bmatrix} 0.25 & 0.125 \\ 0.075 & 0.125 \end{bmatrix},\ B = \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0.3 \end{bmatrix},\ B_1 = \begin{bmatrix} 0.1 & 0.025 \\ 0.05 & 0.075 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 0.8 & 0.5 \\ 0.4 & 0.5 \end{bmatrix},\ N_2 = \begin{bmatrix} 0.2 & 0.125 \\ 0.1 & 0.125 \end{bmatrix}, C_1 = C_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T,$$

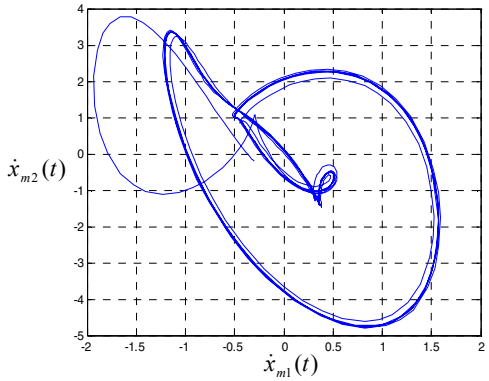
$$C_3 = \begin{bmatrix} 1 & 0.5 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}.$$

The delays $r(t) = d(t) = (1-e^{-t})/(1+e^{-t})$ are time-varying and satisfy $0 \le r(t) = d(t) \le 1$ and $\dot{r}(t) = \dot{d}(t) \le 0.5$. For simulation purpose, a uniformly distributed random signal with minimum and maximum -1 and 1, respectively, as the

disturbance is imposed on the response system. With the above parameters, the neutral master-slave systems (1) and (2) exhibit chaotic behaviours such the $x_{m1}-x_{m2}$ and $\dot{x}_{m1}-\dot{x}_{m2}$ planes with $\xi(0)=col\{0.4,0.6,-0.3,-0.2\}$, $\zeta(0)=col\{0.8,-0.7,0.1,0.1\}$, respectively, are shown in Fig. 1.



(a)



(b)

Figure 1. The phase trajectories: a) $x_{m1}-x_{m2}$ plot and b) $\dot{x}_{m1}-\dot{x}_{m2}$ plot.

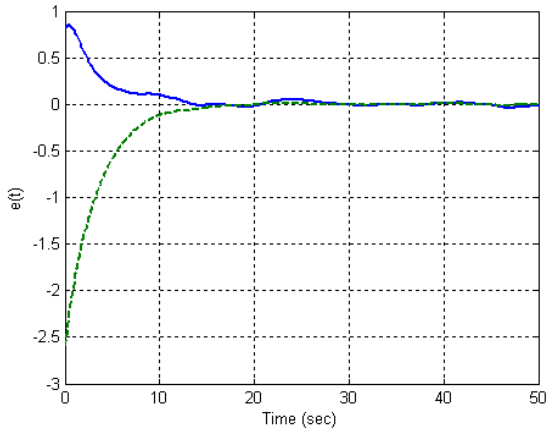


Figure 2. The synchronization errors: a) $e_1(t)$ (solid line) and b) $e_2(t)$ (dashed line).

It is required to design the control law (5) such that the closed-loop system is asymptotically stable and satisfies the H_∞ performance measure. To this end, in light of Theorem 2, we solved LMI (21) with the disturbance attenuation $\gamma=0.2$ and obtained the following control gains by using Matlab LMI Control Toolbox

$$K=\begin{bmatrix}8.9681 & -9.0207 & 37.1101 & -30.0309\end{bmatrix},$$
$$K_r=\begin{bmatrix}-0.0250 & 0.1896 & 0.5152 & -2.1808\end{bmatrix}.$$

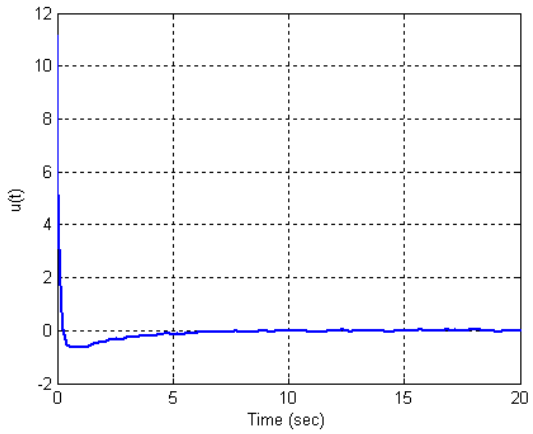


Figure 3. Time-response of the control law for system.

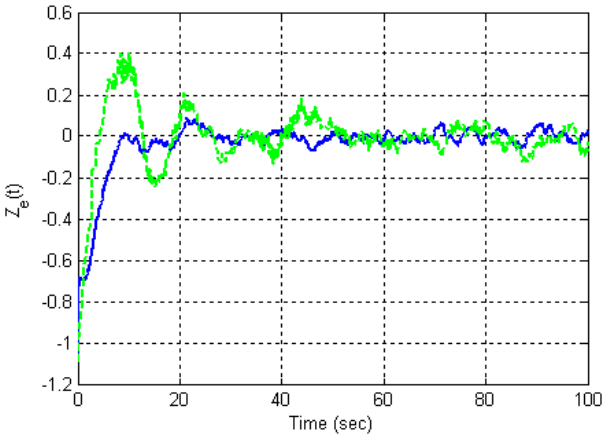


Figure 4. Comparison of the controlled outputs: a) closed-loop system (solid line) and b) open-loop system (dashed line).

Now, by applying the delayed state feedback controller (5) with the parameters above, the synchronization error between the drive system and response system, i.e. $e(t)=x_s(t)-x_m(t)$, is shown in Fig. 2. It is seen that the synchronization errors $e_1(t)=x_{s1}(t)-x_{m1}(t)$ and $e_2(t)=x_{s2}(t)-x_{m2}(t)$ converge to zero. The curve of output-feedback control is also shown in Fig. 3. To observe the H_∞ performance, the response of the controlled output, i.e.,

$z_e(t)$, is depicted and compared with the output signal in the open-loop system under the disturbance in Fig. 4.

V. CONCLUSION

This paper presented the H_∞ synchronization problem of the master and slave structure of a second-order neutral chaotic system with time-varying delays. Delay-dependent sufficient conditions for the design of a delayed output-feedback control were given by Lyapunov-Krasovskii method in terms of an LMI. A controller guaranteeing asymptotic stability, and H_∞ synchronization of the master and slave structure using some free weighting matrices was developed directly instead of coupling the model to a first-order neutral chaotic system.

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