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ORIGINAL PAPER

The professional practice of designing tasks: how do pre-service early childhood teachers promote mathematical processes in early algebra?

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Abstract

Spanish educational curriculum adopts a mathematical process-based approach, which encompasses problem solving, reasoning and proof, communication, connections and representation. A fundamental role in the integration of these processes in mathematics teaching is played by teachers' professional practice of designing tasks. According to this, our aim is to analyze the ways in which pre-service early childhood teachers understand the mathematical processes in the professional practice of designing early algebra tasks and to identify how they intend to promote these processes through the tasks. Content analysis techniques were used to examine the designed tasks. To illustrate the data analysis and results, six tasks are presented. As a result, pre-service early childhood teachers associate problem solving with challenges and questions. They understand problems as unfamiliar situations but ignore the relationships between students and tasks. Moreover, they do not encourage exploration of phases of problem solving and tend to use strategies more suitable for routine tasks. Communication is identified in all the tasks designed, encouraging interaction and discussion. However, only one task explicitly promotes mathematical language. For reasoning and proof, pre-service teachers begin to use questions to elicit explanations and justifications, but do not encourage verification strategies and various modes of reasoning. The process of connections is only present in one task, reflecting the fragmented nature of mathematics teaching. We conclude that the professional practice of designing mathematical tasks is a powerful in teacher education. However, training programs should place greater emphasis on the meaningful use of mathematical processes.

Keywords Design of mathematical task · Mathematical processes · Early algebra · Pre-service teachers · Early childhood education

1 Introduction

The National Council of Teachers of Mathematics (NCTM, 2000) proposes organizing mathematics education into content standards (numbers and operations, algebra, geometry, measurement and data analysis and probability), which describe the content that students should learn; and process standards (problem solving, reasoning and proof, communication, connections and representation), which highlight how the content is acquired and used. In Spain, the education law has opted for a curriculum inspired by the standards proposal of the NCTM, starting from early childhood education and continuing through primary and secondary schooling stages. However, several shortcomings have been identified in the Spanish early childhood curriculum in relation to mathematical processes, suggesting a dilution of their presence. For example, problem-solving and connection processes are mentioned in general terms. There is also an omission of mathematical argumentation, an aspect associated to the process of reasoning and proof. The communication process has a significant presence, while mentions of representation in mathematics are notably scarce (Alsina, 2022). In this sense, more attention should be paid to the inclusion of mathematical processes in early childhood education.

The mathematical process-based approach assumes that in addition to know mathematical contents, it is important to understand how to apply mathematical principles and values in different contexts (e.g., Jackson, 2021; NCTM, 2000; Niss, 2002). In this regard, the Council for the Curriculum, Examinations & Assessment (CCEA, 2020) points to the

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utilization of mathematics as one of the cross-cutting skills that constitutes the core of the curriculum. To encourage students to use mathematical knowledge in a variety of contexts and in different ways it is essential that teachers promote mathematical communication, management of information, critical thinking, problem solving and decision making. To this end, the professional practice of designing tasks is an important factor in teaching during initial training, which should include mathematical processes. For example, to promote the use of mathematics, teachers should propose tasks that relate to real-life situations and involve activating processes of reasoning, communication and/or representation, so that students can transfer their knowledge, if necessary, to other contexts.

In this framework, it should be noted that student learning occurs as a consequence of working on tasks that have been purposefully selected or designed by teachers (Sullivan et al., 2015). Furthermore, student reasoning is influenced by the properties of these tasks (Lithner, 2017). Consequently, it is necessary to emphasize the significance of the tasks designed and proposed by teachers. Through task design, teachers recognize the mathematics they teach as a coherent set of meaningful ideas (Thompson et al., 2007), which in turn leads to an expansion of teachers' knowledge and a better understanding of how to promote student learning. In this work, we focus on pre-service early childhood teachers' learning of the professional practice of task design. According to this, the aim of this article is to analyze the ways in which pre-service early childhood teachers understand the mathematical processes in the professional practice of designing early algebra tasks and to identify how they intend to promote these processes through the tasks.

1.1 Literature review

Early algebra has recently been incorporated into the early childhood curriculum (e.g., Australian Curriculum, Assessment and Reporting Authority, 2020; Ministerio de Educación [MINEDUC], 2018; Ministry of Education, Republic of Singapore, 2013; NCTM, 2000). This perspective is intended to introduce the development of algebraic thinking from early ages (Carraher & Schliemann, 2007; Kaput et al., 2008) aiming to ensure a better understanding of mathematics in later educational stages (Cai & Knuth, 2011). The Spanish early childhood curriculum, while including some content associated with early algebra, omits essential topics such as patterns and change (Alsina, 2022).

So far, studies of early algebra during childhood suggest that the development of algebraic sense at an early age is related to an understanding of the relationships of classification, ordering and correspondence, patterns of repetition and the notion of change in a broad sense as a transformation using an operator. Relationships of classification, ordering and correspondence emerge from the physical knowledge that children acquire in the early years through the manipulation of objects. Thus, children interact with the environment considering characteristics or attributes external to them, which provides an insight into relational thinking (Lenz, 2022). Repetition patterns contribute to the development of mathematical abstraction, by providing opportunities to observe and verbalize generalizations. These includes pattern duplication tasks, finding missing elements in a sequence, extending a sequence, building the same pattern with different elements, identifying the repetition unit, and inventing a pattern (Clements & Sarama, 2009; Lüken & Sauzet, 2020; Rittle-Johnson et al., 2013; Wijns et al., 2019). These tasks contribute to the development of skills related to recursive and functional thinking. Finally, the notion of change and the relationships that are established by change, which are addressed within the framework of functional thinking (NCTM, 2000; Warren & Cooper, 2005) can provide a basis for applying mathematics and understanding the world.

Additionally, a range of studies explore fundamental aspects of teacher training for teaching early algebra and teachers' knowledge of this content. Hohensee (2017), for instance, examines the ideas and challenges experienced by pre-service elementary teachers when exploring early algebraic reasoning. After participating in a semester-long content course, pre-service teachers transitioned from their existing knowledge of formal algebra in high school to new insights into early algebra. They gained comprehension about the operational and relational meanings of the equal sign, and an understanding of the informal representation of unknown and variable quantities without the use of algebraic symbols. Despite these advancements, conceiving algebra from a functional standpoint and engaging in pre-symbolic algebraic reasoning remains a conceptual challenge for them. Ferreira et al. (2022) also expose a formative process in which in-service elementary teachers gained new understandings of algebraic thinking, particularly generalization and regularities. During this formative process, teachers bring into play mathematical and pedagogical knowledge, and develop an understanding of how to work with early algebra in the classroom.

In the study of Gasteiger et al. (2020), the knowledge of 149 pre-service and in-service early childhood teachers was analyzed. Using the model proposed by Gasteiger and Benz (2018), the authors present an instrument that provides a suitable first approach to measure early childhood teachers' mathematical pedagogical content knowledge, while also addressing their implicit knowledge. In relation to early algebra, one situation in the test refers to a geometric pattern. In the early childhood teachers' responses, the wrong answers could be linked to a misunderstanding of the term 'regular pattern'.

On the other hand, Zapatera and Quevedo (2021) study the degree of algebraic knowledge of 106 pre-service primary school teachers at the beginning of their training. The authors introduced two real-life situations that are usually presented when teaching early algebra. Later, they asked the pre-service teachers using these situations to design tasks to foster the development of algebraic thinking in primary students. The results revealed the scarce algebraic knowledge of pre-service teachers, since most of the participants proposed tasks that could be solved numerically or arithmetically but ignored the algebraic nature of arithmetic and prevented students from making connections between arithmetic and algebra.

1.2 Research interest

Task design is a specific professional practice that teachers should develop. This design could involve the creation of entirely new tasks or the adaptation of existing tasks. Ron et al. (2013) state that engaging in the full process of design tasks is challenging for teachers and that they usually "make choices about existing tasks in textbooks, rarely with some (minor) modifications" (p. 641).

On the other hand, Pepin et al. (2013) reveal that working with tasks is an interactive process between the teacher and the resource. When teachers adapt and appropriate a task, they improve their mathematical knowledge and their mathematics-pedagogical design capacity (Pepin, 2015). In this sense, teachers' work with tasks enhances their knowledge both individually in lesson preparation and collectively in professional development sessions and other interactions with colleagues (Jones & Pepin, 2016). This collective dimension is an important aspect of teachers' professional development and design capacity building (Gueudet et al., 2013). Considering the above, and the fact that task design is an activity addressed in teacher training programs, it is interesting to inquire the tasks designed by pre-service teachers; in this case, pre-service early childhood teachers, given the early algebra content addressed in this research.

The objective of early algebra tasks is to facilitate the observation of mathematical patterns, relationships, and structures. These tasks aim to promote different modes of thinking - recursive, relational and functional - that attend to the structure underlying mathematics (Blanton & Kaput, 2005). Then, from early algebra, the development of algebraic thinking is promoted through express, justify and reasoning with generalization (Stephens et al., 2015). This content plays a relevant role in the development of mathematical skills (Ayala-Altamirano et al., 2022; Pinto & Cañadas, 2021). Moreover, tasks used by teachers to teach early algebra should serve a double purpose. On the one hand, by fostering the development of algebraic thinking. On the other hand, by motivating teachers to enable students to think deeply about relationships, patterns and change (Twohill et al., 2019). In this sense, early algebra tasks could be associated with the different mathematical processes. The preceding observations suggests that the design of early algebra tasks provide a conducive context for incorporating mathematical processes in early childhood education.

Against this backdrop, we pose the following research question: How do pre-service early childhood teachers promote mathematical processes when designing algebra tasks? To address this question, we asked a group of pre-service early childhood teachers to design tasks of early algebra. Subsequently, an analysis of the tasks will be conducted, considering the mathematical processes that pre-service teachers indicated they intend to promote with the task and their understanding of these processes.

2 Conceptual framework

2.1 Mathematical processes in early childhood education

Mathematical processes highlight the ways of looking for and applying mathematical knowledge, making them crucial in the teaching and learning of mathematics at all school levels. Several authors emphasize the importance of fostering reasoning, justification and argumentation in early childhood education (Cornejo et al., 2021), as well as problem solving (Li & Disney, 2023; Vessonen et al., 2023), the use of representations (Langrall et al., 2008), the communication of children's mathematical ideas (Ginsburg, 2009) and the establishment of connections (NCTM, 2000). In this sense, early childhood teachers' understanding of these mathematical processes supports the professional practice of designing and implementing tasks that facilitate the construction of mathematical knowledge through the development of such processes.

Regarding the problem-solving process, in different research traditions, a problem is usually defined as a new situation, challenge, or type of task that the student does not know beforehand how to solve or provide an answer to (Schoenfeld, 2007). This suggests that the status of a problem is not defined by the number of steps involved or the technical complexity of the task. Consequently, the question of whether a task represents a problem is dependent on the interaction between the task and the solver. As Lester (2013) observes, this conceptualization represents a fundamental aspect of problem-solving research. In order to establish that a task constitutes a problem, it is necessary to ascertain (or hypothesize) that (1) the student lacks familiarity with the principal components of the solution and (2) the student possesses the requisite knowledge to devise their own solution. In this vein, the process of solving a problem can be described as the looking for and applying of appropriate strategies that allow the solver to obtain a successful outcome by responding to the demands of the problem. These strategies can take the form of techniques, inventions, criteria, or rules which are distinguished from routine methods. Also, the work with problems, including the phases to problem solving and problem-solving strategies (Pólya, 1945; Schoenfeld, 2007) is deemed pertinent in early ages (Alsina, 2014; Li & Disney, 2023). Considering the above, to engage students in problem solving, early childhood teachers should develop the mathematical disposition to posing problems. This should extend from the everyday routines of the classroom to mathematical situations that provide students with opportunities to apply, extend, and build new mathematical knowledge (Vessonen et al., 2023).

On the other hand, participating in reasoning, justification and argumentation are core elements of mathematics education in early childhood (Ginsburg, 2009). Actions such as conjecturing, explaining, discovering, proving and checking are related to the development of mathematical thinking and the reasoning and proof process. Bransford et al. (1999) pointed out that young children can begin to reason based on their experiences. Then, reasoning is developed when they are prompted to formulate conjectures and seek evidence that either confirms or refutes them. Consequently, although reasoning is mainly inductive at early ages, students should gradually internalize deductive reasoning (Alsina, 2014). Following the above, the posing of good questions (Sullivan & Lilburn, 2002) or purposeful questions (NCTM, 2014) can facilitate the promotion of reasoning (and thus communication) within the mathematics classroom. The use of these questions allows students to explain, justify, or argue their actions, proposals, and results. Instead of validating the results, students are encouraged to verify them. So, regarding the reasoning and proof process, the importance of strategies to promote explanations, arguments and justifications, strategies to check actions and answers and reasoning methods (inductive and deductive) are emphasized.

Also, the NCTM (2000) propose a teaching focus on the development of the organization and consolidation of mathematical thinking through communication. This mathematical process encompasses the consistent and clear expression of mathematical thinking to peers, teachers, and others; the analysis and evaluation of the mathematical strategies and thinking of others; and the use of mathematical language to express mathematical ideas accurately. In this regard, questions stand out as one of the most suitable mediations precisely because they encourage children to progress from the first levels of awareness about what one already knows or can do, toward higher levels where they catch glimpses of how best to advance their learning (Alsina, 2014). In this sense, the communication process refers to the interactions between students and between students and teacher with the aim that students achieve to communicate their mathematical ideas. In this interactions, mathematical knowledge is either deconstructed, co-constructed, or reconstructed. The mathematical vocabulary to be used, introduced, or consolidated in a mathematical situation is also considered in this process.

Another relevant mathematical process is making connections, which enables the development of a consistent system of mathematical knowledge, and enhance understanding of the broad applicability of mathematics (Kinnear et al., 2018). Alsina (2014), proposes three main types of mathematical connections for early ages based on ideas of NCTM (2000). These are: (a) connections between different blocks of mathematical content (intradisciplinary connections); (b) connections between mathematics and other areas of knowledge (interdisciplinary connections); and (c) connections between mathematics and the environment. In accordance with the NCTM (2000), it is important to adopt a teaching approach from the age of three that promotes progressive work around three key areas: the recognition and use of connections between mathematical ideas; an understanding of how mathematical ideas are interconnect and build upon one another to produce a coherent whole; and, finally, the recognition and application of mathematics in non-mathematical contexts.

On the representation process, Langrall et al. (2008) point out the importance of manipulatives, mental models, symbolic notation, tables, graphs, number lines, stories and drawings to organize and convey thinking. In addition, other forms of representations such as oral or motor expression are considered. Through interactions with these different types of representations, children develop their own mental images that allow them to further their learning of mathematics. From this point of view, the NCTM (2000) proposes that teaching from an early stage should be gradually focused on: creating and using representations to organize, record, and communicate mathematical ideas; the selection, application and translation of mathematical representations to solve problems; and the use of representations to model and interpret physical, social, and mathematical phenomena. Then, the representation process encompasses the use of manipulative materials to represent mathematical ideas, as well as the use of different forms of representations – including concrete (e.g., drawings), pictorial (e.g., signs) and/or symbolic (conventional notation) – in addition to other forms of external representations (oral, motor). Representations are also essential for the communication of mathematical ideas, which these two processes are intrinsically linked.

2.2 Design of mathematical tasks

In this study, a mathematical task is considered as the information that drives the work with students, including representations, context, questions and instructions (Sullivan et al., 2012). As per Margolinas (2013), a mathematical task "affords opportunity to encounter mathematical concepts, ideas, strategies, and also to use and develop mathematical thinking and modes of enquiry" (p.10). So, mathematical tasks should promote high levels of thinking (Stein et al., 2009) because through of the challenging tasks that mathematics teachers propose, effective mathematical learning is achieved (NCTM, 2014). Mathematical tasks are then a source of opportunity for students and a challenge for teachers (Sullivan et al., 2010).

Ron et al. (2013) state that task design is usually a hierarchical process that considers the stages of stating the goal(s) and connecting the task to the goal(s), designing a generic task that addresses these goals and, in some cases, carefully choosing the specific examples to "plug into" the generic task. The design of a mathematical task also includes "the necessary materials, task sequences and advice about effective choices, and detailed pedagogic advice about ways of working, verbal interventions, likely misconceptions and possibly extensions" (Margolinas, 2013, p.10). Another aspect to consider in the design of tasks is to try to anticipate the questions that learners may ask, and to have a general idea of the answers that support the task design (Ainley et al., 2005).

In summary, the literature provides various guidelines and elements that should be incorporated into the design of a mathematical task. Following these points of view, we consider that the professional practice of designing mathematical tasks begins with the definition of the objectives and their link with the proposed task and continues with the incorporation of elements of a pedagogical-disciplinary nature.

3 Methodology

3.1 Participants and context

The participants in this study were 105 Spanish pre-service early childhood teachers from the University of Girona (Catalonia, Spain). The participants were selected based on their availability during the second-year course named *Learning Mathematics*. This is the only core Mathematics Education course that participants take during their training process. The *Learning Mathematics* course deals with the curricular design of the mathematics of early childhood education (3–6 years old), the analysis of the content and mathematical processes that comprise the curriculum, and the developing methods for adapting mathematics education for young children. The course delves into the main aspects of the development of mathematical competence in early childhood education: task design from different contexts (real contexts, manipulatives, games, applets, etc.) and teaching practices through mathematical processes. Mathematical content is also presented, in the following order: early algebra, numbers and operations, geometry, measurement, statistics and probability. The research was carried out in the first content standard, corresponding to early algebra.

In terms of early algebra, the course looks at recognizing attributes to establish relationships, repetition patterns and describing change, detailing the progression of learning for 3-, 4- and 5-year-olds. We consider it is relevant to conduct research with this group of pre-service teachers because *Learning Mathematics* is the only second year course that deepens mathematics teaching and learning, over a of 3–5 years span. In addition, there is no other course in the training of pre-service teachers that focuses on teaching practices through mathematical processes.

3.2 Data collection

Over the course of two 90-minute class sessions, the participants were asked to work in small groups (3–4 pre-service teachers) to design mathematical tasks to support the teaching of early algebra. Each working group is asked to design three early algebra tasks for 3-, 4-, and 5-year olds (one task for each age) following the instructions below for each task: (1) selecting an algebraic content to be developed in the task, taking into account the age of the students; (2) indicating the mathematical processes promoted by the task; (3) describing the instructional context to be address and the management more appropriate, for example, how students could be grouped, how the task could be introduced and how to the task could be handle or carry out in the classroom; (4) solving the task; (5) posing questions to promote students' understanding of the task; and (6) indicating the mathematical language to be promoted in the task.

According to the stages proposed by Ron et al. (2013), the items 1, 2 and 6 are related to setting goals and connecting the tasks to the goals. The items 3, 4 and 5 are related with the design of a task that address these goals. Also, it should be noted that the working groups designed the tasks during two class sessions, under the professor's supervision. They themselves proposed the task goals based on the knowledge gained and the instructions received. A total of 32 working groups were formed, mostly consisting of three pre-service early childhood teachers.

3.3 Data analysis

To organize the data, an analysis was conducted on the 96 tasks proposed by the pre-service early childhood teachers' working groups. The level of cognitive demand of the tasks was not investigated, however, we ensured that all the mathematical tasks proposed by the pre-service teachers were related to the learning of early algebra, considering the objectives and mathematical contents presented in the *Learning Mathematics* course based on the levels of 3-, 4- and 5-year-olds. An examination was conducted to ascertain the coherence between the selected algebraic content (item 1) and the solution of the mathematical task and its proposed management (items 3 and 4). Also, a revision of the responses to items 2, 5 and 6 to ensure that the mathematical processes are incorporated into the designed tasks was developed.

In alignment with the aim of this study, a qualitative methodology was employed. A content analysis technique (Krippendorff, 2013) was implemented to analyze the responses of the groups of pre-service teachers by designing the tasks. In this regard, a comprehensive and detailed examination was conducted of the statements made by pre-service teachers regarding their understanding of mathematical processes and the ways in which they intended to promote these processes through tasks. The aforementioned statements constituted the unit of analysis in the present study.

Six tasks that illustrate (in a more comprehensive manner) the data analysis and the result are exposed. The tasks were selected based on the following criteria: (1) the age of the students to whom the tasks are addressed (3-, 4- and 5-years old); (2) the early algebraic content involved, considering tasks that deal with the recognition of relationships and attributes, patterns of repetition, and qualitative and quantitative changes).

4 Results

Tasks for different ages (3, 4 and 5 years old) and tasks representative of the different algebraic contents addressed in Early Childhood Education are presented below. The tasks address relationships based on the recognition of attributes; for example, the grouping of elements based on the identification of their properties and the comparison of elements. Sequence tasks with repetition patterns were also presented by, for example, asking students to extend a sequence or create a pattern from given elements. Finally, tasks in which qualitative and quantitative changes must be recognized and applied were observed.

4.1 Tasks designed for 3-year-olds: understanding of problem solving as challenges and communication as dialogue

A task designed by a group of pre-service childhood teachers to work on relationships based on attribute recognition with 3-year-olds is presented in Fig. 1. The task is designed to teach students to classify objects using color as a qualitative criterion.

As illustrated in Fig. 1, the pre-service teachers design a task focused on classifying elements. The discussion proposed in the task, addresses the relationships between the elements to be classified by recognizing the different sensory qualities and attributes. The group of pre-service teachers

Colored fish

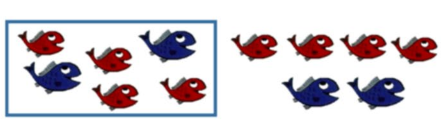
1. Content Relationships from attribute recognition (3 years). **2. Mathematical processes** Problem solving through the posing of a challenge and communication through teacher-pupil dialogue and the development of mathematical vocabulary.	**4. Solution of the mathematical task** To classify the fish, the children should be able to establish different criteria according to the recognition of their attributes. Each working group establishes different criteria for making classifications of the goldfish, from the comparison they can determine that groups of fish can have different numbers of elements. To answer the last question, the children should argue that more than one criterion can be considered to classify the fish in the tank, e.g. red fish with a rough texture.
3. Management of mathematical tasks The children are arranged into groups of six students. A fish tank containing various fish with different qualities (color, texture, size, etc.) are showed. The teacher stars a dialogue about the qualities observed in the manipulative material: What are the fish we have in our tank like? For example, If I take 4 fish, what characteristics they have on common? What differences they have? 	**5. Questions proposed in the mathematical task** Later, a challenge is proposed to each group. For example, "taking six fish and form a group" or "taking 8 fish and arrange them into small groups". The teacher asks: - What groups can you form with the fish you took out of the fish tank? - What criteria did you use to form the groups? Did all the groups of students use the same criteria? - Do all groups of fish have the same number of elements? - Can the red and coarse fish be in the same group? Why? **6. Mathematical language** Classifying, grouping, equal, different.

Fig. 1 Colored fish

asserts that the task promotes the *problem-solving process*. In this case, classifying fish according to a color criterion is perceived as a challenging task for students, thereby describing a problem.

Moreover, the pre-service teachers posit that the task would facilitate the *communication process* in two distinct ways. In the first instance, the questions related to the classification of elements are designed to promote interaction between the students and between the teacher and the students, thereby enhancing comprehension of the algebraic content involved through communication. On the other hand, according with the instructions regarding the task design, the group of pre-service teachers highlight the terms equal, different, grouping and classifying as part of the mathematical language that the students could acquire while developing the task.

In the design of this task, the group of pre-service teachers associates the process of problem solving with the presentation of challenges and the process of communication with the interaction between the students and the teacher. In this sense, both mathematical processes are encouraged by the questions presented in the task.

In Fig. 2, a mathematical task designed by pre-service early childhood teachers to engage three-year-old children in exploring relationships of ordering and classification is presented.

In this task, pre-service teachers link the *problem-solving process* with questioning. Students are encouraged to find and implement a strategy to solve the task, thereby pre-service teachers understand a problem as a question that needs to be answered to provide a solution. Furthermore, since the context of the task is a children's story, the pre-service teachers state that the task promotes the *making connections process*. Specifically, they consider that using a story as a setting for the task promotes interdisciplinary links between mathematics and children's literature.

Also, the pre-service teachers highlight that the dialogue between the teacher and students, established through asking and answering questions, could facilitate the expression of the students' mathematical ideas related to setting the appropriate order and criteria for classifying elements of the story. This dialogue would promote the *communication process*, allowing for the co-construction of mathematical knowledge, emphasizing the recognition of attributes, and enabling the reconstruction of such knowledge in case of student errors.

In designing this early algebra task, the group of pre-service teachers associates the *process of problem solving* with providing a solution to a task within a given context, without delving deeply into whether this task accurately represents a challenge for students. The *communication process* is understood as an interaction between students and teachers, which is intended to be generated through the questions

Folktale: Goldilocks

1. Content Relationships from attribute recognition (3 years). **2. Mathematical processes** Problem solving through open questions; communication through dialogue between teacher and pupils; and connections between mathematics and children's literature via a folk tale.	**4. Solution of the mathematical task** It is expected that the children will be able to order each of the objects based on their sensory qualities such as size and shape.  To establish classifications, the teacher can explain to the children that each bear corresponds to certain objects. Therefore, they can classify all the objects as big, medium, or small. Throughout the lesson, the teacher interacts with the children to gather responses to questions and address any errors or doubts raised by the students.
3. Management of mathematical tasks To contextualize the task, the teacher should begin by reading the story of Goldilocks. To assist students in understanding how to approach the activity, the teacher should summarize the story, focusing on the characters and objects within it. All the objects mentioned in the story are described as being either big, medium, or small in size. Following this introduction, the teacher distributes different cards to the students. A total of 15 cards depicts pictures with the different objects in the story and their possible sizes. Subsequently, questions are posed to the children to guide their understanding of ordering and classification relationships.	**5. Questions proposed in the mathematical task** - How can we arrange the bears, chairs, beds, and shoes from largest to smallest? How can we arrange them from smallest to largest? - Can we group the different cards? How many groups of elements can we create? What criteria are you using to classify them? How do you go about classifying them? **6. Mathematical language** Classifying, ordering, grouping.

Fig. 2 Folktale: Goldilocks

that are presented and the cards as visual aids to facilitate the task. These elements are proposed to enable children to establish the appropriate ordering and criteria for classifying the elements of the story.

4.2 Tasks designed for 4-year-olds: understanding of questions as a way of linking mathematical processes

Figure 3 shows the components of a task designed for 4-year-olds. The purpose of the task is to teach students to extend sequences of repeating patterns.

In this task, the group of pre-service teachers posits the promotion of mathematical *process of problem-solving*. The task features an interactive game drawn from an educational platform named JClic. The resource allows students to make predictions about potential solutions to extend the sequence, thereby providing responses to the various challenges posed. In the tasks, pre-service teachers provide guidance to students, who should observe the sequence and hypothesize about the elements that need to be added to extend it. The identification and extension of the sequence is considered by the pre-service teachers as a *problem-solving* strategy.

In addition, the pre-service teachers point out that with this task, students could add new words to their mathematical vocabulary related to early algebra, such as series, pattern, or repetition. Thus, to facilitate students' comprehension and internalization of this algebraic language, the pre-service teachers suggest questions that include these terms. In this context, the pre-service teachers are seeking to further the *communication process*.

In designing this task, the group of pre-service teachers associates *problem solving* with a challenge and considers strategies for its resolution. However, they do not assess whether the strategies that they expected students employ are specifically designed for problem solving. Regarding to the *communication process*, the mathematical vocabulary that can be incorporated into the tasks development and the questions that are formulated allow for the inclusion of these new words.

Figure 4 shows a mathematical task that aims for students to describe and apply qualitative and quantitative changes.

The task is designed to introduce 4-year-old students to the concept of change through a game of dominoes. The game consists of taking a piece, describing its characteristics, and then placing pieces on the condition that a qualitative or quantitative characteristic change between them. According to the pre-service teachers, this task would involve the *problem-solving process*, since to participate in the game, the students must find a solution that is determined by the previous piece.

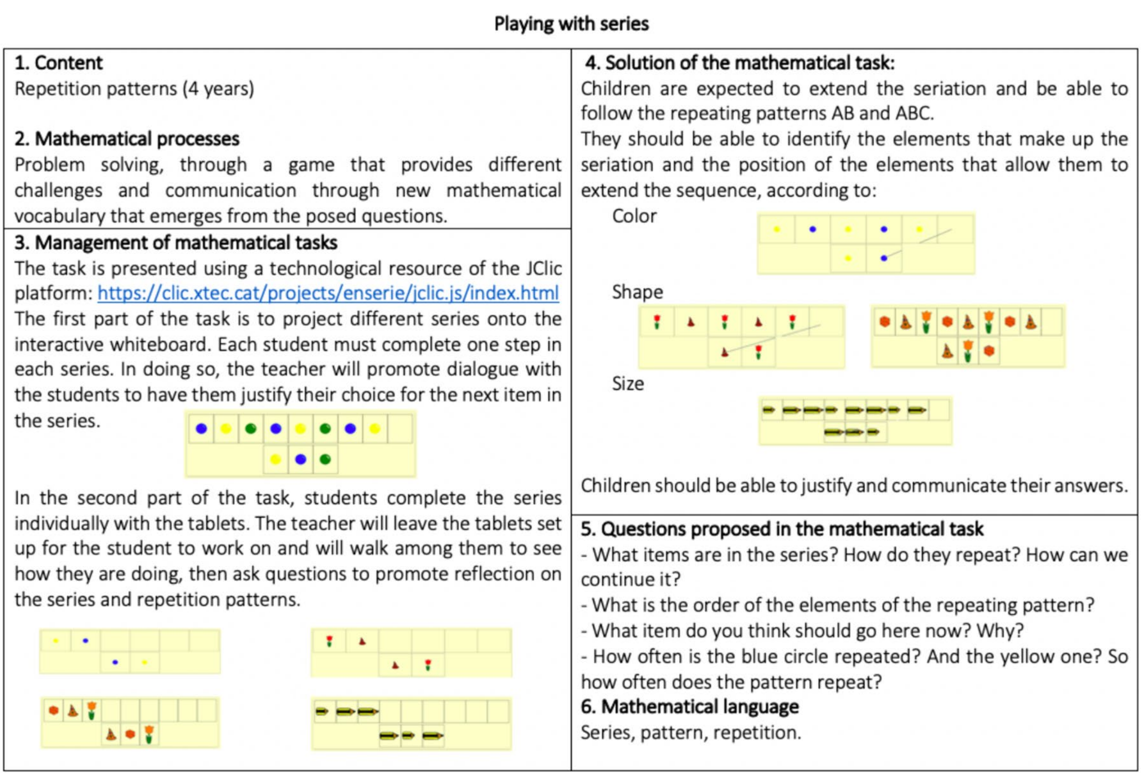


Fig. 3 Playing with series

Ladybird dominoes

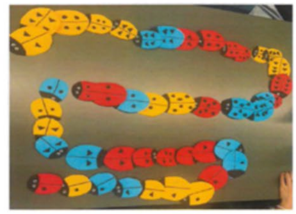
1. Content Qualitative and quantitative changes (4 years). **2. Mathematical processes** Problem solving, because in each round the students have to look for the solution that is determined by the previous piece. Reasoning and proof, because the students have to look for the evidence why they can choose a certain piece or not, and if they make a mistake, they have to ask themselves why. Communication, because the task is individual, but the pupils can talk to each other to facilitate the game and help each other.	**4. Solution of the mathematical task** To respond to the task and participate in the game, children must apply qualitative and quantitative changes, considering the different attributes of the pieces. 
3. Management of mathematical tasks A domino is prepared with 46 pieces with different ladybirds, classified according to the following criteria: a) color (red, yellow and blue), b) size (big and small), c) number of spots (1, 2, 4 and 6), d) shape of the spots (triangles and circles). The pieces are distributed to all students. One child chooses a piece and starts the game, the next one must continue with a domino piece that changes only one attribute. For example: a large yellow ladybird with two triangles is selected and then a large yellow ladybird with four triangles (the number of triangles has been changed). The first player to run out of pieces is the winner and the game is over.	**5. Questions proposed in the mathematical task** - What does your ladybird look like? How many spots does your ladybird have? What is the shape of your ladybird's spots? - What is the difference between your ladybird and your partner's ladybird? - What attributes have changed between your ladybird and your partner's ladybird? - Do the rest of the partners agree with what has been explained? **6. Mathematical language** Attributes, qualitative change, quantitative change.

Fig. 4 Ladybird dominoes

In this sense, pre-service teachers associate problem solving with asking and answering questions to find a solution.

On the other hand, pre-service teachers state that the designed task promotes the *process of reasoning and proof*, since through the questions proposed by the teacher (e.g., what attribute changes between your ladybug and your partner's ladybug?), students must check the result of their action, i.e., whether the selected piece meets the condition of the game. In this vein, the pre-service teachers point out that the management of the domino game also involves the *communication process*, since the interaction between the students proposed in the game is a tool to promote the exchange and communication of mathematical ideas.

In this task, it was observed that the group of pre-service teachers associated *problem solving*, *reasoning*, *and proof* with the formulation of questions to find a solution. Furthermore, the *communication process* is associated with the interaction between students and the teacher.

4.3 Tasks designed for 5-year-olds: Understanding of problem solving as the sequencing of steps and communication as the integration of mathematical language

In the mathematical task presented in Fig. 5, pre-service teachers incorporate the use of a technological tool such as Bee-Bots to engage students in extending sequences and inventing patterns of repetition. The task is supported by the visual aid of cards representing AAB, ABB, and ABC patterns.

In this task, pre-service teachers propose guiding students through the work to be done and helping them to understand the indications presented in the instruction cards regarding the performance of Bee-Bots movements. In this vein, pre-service teachers indicate that they promote the *problem-solving process* linked to the formulation of questions during the use of the programmable educational robot.

Creating patterns with Bee-Bots

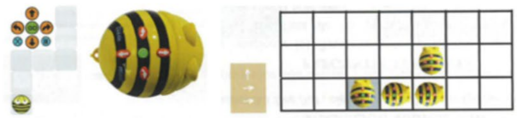
1. Content Patterns of repetition (5 years) **2. Mathematical processes** Problem solving, based on the guidelines given by the teacher, using the instruction cards to perform movements with the educational robot and the questions asked to develop a series of steps to create patterns and extend the sequence. Communication, through dialogue and discussion between the students and the teacher when working with repetition patterns.	**4. Solution of the mathematical task** The task involves children creating repeating patterns of three elements (AAB, ABB and ABC) and then extending the series with the help of the Bee-Bots. The children have to press, for example, the right arrow twice and the up arrow once. 
3. Management of mathematical tasks The task is implemented in small groups of five children, who interact and make their own decisions in creating repeating patterns. Each group is given a Bee-Bots educational robot and cards representing a given pattern. The teacher starts with a short explanation of how Bee-Bots work, showing an example with an AAB pattern. Then, using the arrows illustrated on the cards, the children have to create a repeating pattern and then extend the series with the help of the Bee-Bots. The teacher interacts with the different groups, asking effective questions and to support students' use of vocabulary associated with working with AAB, ABB and ABC repetition patterns using programmable educational robots.	**5. Questions proposed in the mathematical task** The development of the activity can be managed through the following questions: - What movements can be achieved with Bee-Bots? - Can we build a different pattern to the one on the card, but with the same movements? What would the path of Bee-Bots look like with this new pattern? - Can you describe the new series in words? - Can you describe the series using symbols instead of words? **6. Mathematical language** Patterns, repeating units, patterns-based predictions.

Fig. 5 Creating patterns with Bee-Bots

Another meaning related to problem solving is the sequencing of steps to create repetitive patterns and, subsequently, the extension of serializations from the pattern created.

On the other hand, according to the pre-service teachers, they try to encourage the *communication process* in the task. This process is promoted by the interaction and discussion generated between the students and the teacher about the central elements of the repetition pattern that are essential for extending and creating sequences (repetition unit and predictions according to the pattern). In turn, pre-service teachers state that this contributes to the incorporation of the word patterns in their algebraic language. For example, in the proposed interaction between teachers and students from different groups, the proposal is to assist them in the use of vocabulary associated with repetition patterns.

In designing this task, the group of pre-service teachers identified two distinct but interrelated approaches to *problem-solving process*. The first of these is the formulation of questions, which is understood as a key component of the problem-solving process. The second is the sequencing of steps that allows for the response to a task based on an instruction. The *communication process*, meanwhile, is understood as the interaction and discussion that is generated between students and teacher, as well as the incorporation of vocabulary linked to algebraic language.

In Fig. 6 the content of the mathematical task designed by pre-service early childhood teachers to work on change with 5-year-olds is presented.

The pre-service teachers state that the designed task promotes the *problem-solving process* by providing a challenge using the changing machine as a manipulative. In the task, students are asked to describe the characteristics of the monster that goes in and out of the machine and to identify the operator who makes the change possible. In addition, the pre-service teachers state that they promote the *communication process* by formulating intentional questions that impact on the description of the characteristics of the monsters to describe their attributes, thus incorporating mathematical vocabulary associated with the application of qualitative and quantitative changes.

During the design of this task, it is observed that the group of pre-service teachers associate problem-solving process with the presentation of a challenge using the change machine. The *communication process* is understood from the interaction between students and teacher, which is generated by intentional questions.

The color monsters

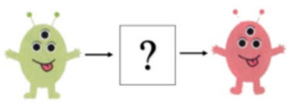
1. Content Qualitative and quantitative changes (5 years) **2. Mathematical processes** Problem solving, through the posing of a challenge that involves the use of the machine to change qualities, and communication, since different questions are posed in the development of the task.	**4. Solution of the mathematical task** It is expected that children begin to identify qualitative changes in relation to color and size, justifying and communicating their ideas. Subsequently, children should make conjectures about the possible monsters that can come out of the quality-changing machine when entering a large blue monster with one eye. In this part of the task, they should apply qualitative and quantitative changes and recognize the characteristics that have changed (e.g. color, size and number of eyes).
3. Management of mathematical tasks We will pose a challenge with the machine for changing qualities designed with a cardboard box and monsters of different colors (green, pink and blue), sizes (small and large) and number of eyes (1, 2 or 3).The teacher explains that a monster goes into the machine for changing qualities and a different one comes out, for example: "If we have a large, green monster with three eyes, and a large, pink monster with three eyes come out, what quality of the monster has changed?"  The students will see the monster that goes into and out of the machine, and will have to determine what characteristic has changed, be it its color, size or the number of eyes.	**5. Questions proposed in the mathematical task** Various questions are posed related to the attributes of the monsters, and based on the dialogue, the students are asked to find answers using the operators: - If a small, blue three-eyed monster goes into the machine and a large, green, three-eyed monster comes out, what changed? What stays the same? - If a large, blue, one-eyed monster goes into the machine, what monster might come out? Why? Are there other options? What are they? **6. Mathematical language** Operator, applying qualitative changes, applying quantitative changes.

Fig. 6 The color monsters

5 Discussion and conclusions

The professional practice of designing early algebra tasks is analyzed in this study with emphasize in the promotion of mathematical processes of problem solving, reasoning and proof, communication, connections and representation.

In the case of *problem-solving process*, the pre-service teachers associate problems with challenges and questions, with no differences depending on the age for which the task was designed: In the two tasks for each age, one associates the problem solving process with a challenge (tasks "Colored Fish", "Playing with Series" and "The Color Monsters") and the other with asking questions or giving answers (tasks "Folktale: Goldilocks", "Ladybird Dominoes", and "Making Patterns with Bee-bots"). The pre-service teachers demonstrate an understanding of the concept of problem relate to an unfamiliar situation (Schoenfeld, 1985). However, there is insufficient evidence that they conduct a comprehensive analysis of the relationship between the task and the learner. This analysis should consider whether the tasks could be considered problems, or whether the learners possess the requisite knowledge to understand the problem and devise a solution.

Another aspect to note is that in early algebra tasks we did not identify that the pre-service teachers encourage students to explore the different phases of problem solving (Pólya, 1945; Schoenfeld, 2007). This is also observed when designing some tasks, there is still no appropriation of the strategies that are characteristic of problem solving, that is, the use of general rules to make progress in unknown situations (Schoenfeld, 1985). In this vein, it was noted that several of the pre-service teachers' strategies are not specific to problem solving for example, classifying items (task "Colored Fish") or identifying the elements that make up the sequence and then extending it (Playing with Series task) are strategies that could be used in solving a routine task. In summary, the analysis shows that pre-service teachers do not yet understand relevant aspects of problem solving, such as the fact that a problem is determined not only by the characteristics of the task but also by the relationship between the task and the student.

Regarding the *communication* process, it is identified in all tasks designed by the groups of pre-service teachers. The communication process is understood to be linked to the interactions among early childhood students and the teacher when discussing resolution processes (e.g., Clements & Sarama, 2009; NCTM, 2000). These interactions arise from posing questions that drive the search for and application of early algebra knowledge. The presence of this process could be influenced by the instructions for designing tasks (Items 5 and 6). However, it is important to note that although the instructions received specify that pre-service teachers should declare the mathematical language promoted by the task (Item 6), only in the task "Playing with Series" does the student group link the communication process with the use of mathematical language to express mathematical ideas precisely (NCTM, 2000).

Related to the mathematical process of *reasoning and proof*, pre-service teachers state that begin to promote it by posing questions for students to explain, argue or justify their actions (task "Ladybirds Dominoes"). In this sense, it seems that they begin to understand that questions are a tool for introducing work with different mathematical processes at early age. However, there are also aspects of reasoning and proof that are not identified in the designed tasks. For example, promoting strategies for checking actions and responses and different modes of reasoning. Authors such as Bransford et al. (1999) have already pointed out the importance of formulating conjectures and carrying out checks. Organizations such as the National Council for Curriculum and Assessment (NCCA, 2014) underscore the need to help children understand their thinking and help them express it to others. According to the above, there is a need to go deeper into these aspects in initial teacher training, considering the input of some authors on the importance of these components of the reasoning and proof process (Blanton et al., 2018; Pinto & Cañadas, 2021).

Also, we highlight the presence of the *connections* process in one of the six tasks analyzed (task "Folktale: Goldilocks"). This result is of especial interest given that both Spanish and international curricular guidelines, propose an integrated teaching, since compartmentalized knowledge cannot address complex and global issues (Perignat & Katz-Buonincontro, 2019). In Spain, for example, mathematical competence is part of an integrated STEM competence called Competence in Mathematics and Competence in Science, Technology and Engineering, which assumes that children are initiated in logical-mathematical skills and take the first steps towards scientific thinking through play, manipulation, and the performance of simple experiments (Ministerio de Educación y Formación Profesional [MEFP], 2022). One of the main factors behind this lack of promotion of the connection process could be that, even though the early childhood curriculum in Spain breaks with the structure of subjects and is organized into three general areas of learning (MEFP, 2022), the pre-service teachers learned mathematics in an isolated and compartmentalized way in the different content standards, from primary to university. Also, the Spanish early childhood education curriculum does not focus on big mathematical ideas that link numerous mathematical understandings into a coherent whole (Toh & Yeo, 2019). Then, the perspective of intradisciplinary and interdisciplinary connections are barely present making it difficult to consider connections in the design of mathematical tasks.

6 Final considerations

The findings of this article have shown that learning to design tasks to promote the mathematical processes in early algebra is a complex issue for pre-service early childhood teachers. It was observed that the participants in the study were able to incorporate terms associated with mathematical processes, but with a lack of understanding or clarity regarding their meaning. This highlights the necessity for training programs to place a greater emphasis on this facet of the early pre-service teachers professional learning.

On the other hand, the analysis developed in this study only involved the design of early algebra tasks, the first content standard that is addressed in the *Learning Mathematics* course. This factor may have conditioned some of the data obtained, so it will be necessary to develop studies that analyze how mathematical processes are intended to be promoted when the pre-service early childhood teachers design tasks involving other content standards. Thus, it will be necessary to assess the significance of the tasks proposed by pre-service teachers, to deepen the formulation of the tasks designed and to analyze the level of cognitive demand of task. Additionally, it will also be necessary to determine whether the presence of the processes involved in the tasks varies according to the age for which the tasks are designed (3, 4 or 5 years old). Also, the mathematical processes were analyzed exclusively through the professional practice of designing tasks. Although tasks that teachers design can provide evidence of the teachers' perception of learning and teaching mathematics (Thanheiser, 2015) further studies could address the presence of mathematical processes in mathematics teaching practices through research methods, such as the Japanese lesson study, the analysis of class videos, interviews with teachers and others.

Finally, given that algebraic thinking is a general skill that promotes a better understanding of mathematics (Cai & Knuth, 2011), the design of early algebra tasks provides pre-service teachers with experience and knowledge that can be used for the design of tasks from other content standards. Also, the fact that pre-service teachers learn to teach mathematics through processes will encourage them to replace teaching practices based on repetition, memorization and the use of decontextualized exercises with other practices based on thinking and doing mathematics (Alsina et al., 2021), which are some of the keys to the development of mathematical skills from early childhood education onwards. In addition, with the aim of ensuring that teachers acquire the necessary knowledge to promote the development of mathematical processes, it will be necessary to foster the design of pre-service (and in-service) teacher training programs that are organized around the standards of mathematical processes, and not exclusively around content standards, which is currently the most common trend in Spanish universities.

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