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CONDORCET CONSISTENCY AND PAIRWISE JUSTIFIABILITY UNDER VARIABLE AGENDAS*

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We compare the consequences of imposing upon collective choice functions the classical requirement of Condorcet consistency with those arising when requiring the functions to satisfy the principle of pairwise justifiability. We show that, despite the different logic underlying these two requirements, they are equivalent when applied to anonymous and neutral rules defined over a class of domains. The class contains the universal, the single-peaked, and that of order restriction, among other preference domains.

1. INTRODUCTION

We examine the performance of collective choice functions that select one alternative for each admissible *situation*. A situation is a pair consisting of a profile of individual preferences and a subset of alternatives (an *agenda*) from which society may choose. Specifically, we compare imposing the Condorcet criterion with imposing anonymity, neutrality, and pairwise justifiability upon collective choice functions. Pairwise justifiability requires that, if a rule selects alternative x in situation 1 and alternative y in situation 2, there must be an alternative z, and some member of society whose appreciation of z relative to x has increased when going from situation 1 to situation 2 (the appreciation being defined by the agent's preferences in each corresponding situation).

We prove that the set of neutral, anonymous, and pairwise justifiable rules coincides with those that respect the condition of Condorcet consistency on appropriately restricted domains.

There is a significant difference between the properties that we compare. Condorcet consistency has implications for single situations. In contrast, anonymity, neutrality, and pairwise justifiability are conditions on the admissible changes in the social choice when situations change.[1] Pairwise justifiability has implications when either preferences or agendas or both vary.

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[1] For interested readers, Fishburn (1973, 1973b) distinguishes between intra- versus interprofile conditions and presents a more elaborate taxonomy of normative requirements.

What we do is to establish the equivalence, under appropriate conditions, of these two requirements of a rather different nature, giving support to each one of them from the perspective of the other.

Two antecedent papers relating Condorcet consistency with properties on rules requiring outcome comparisons across different preference profiles are by Campbell and Kelly (2003, 2015). In these two papers, the authors prove the equivalence between the majoritarian requirement and strategy-proofness when applied to collective choice functions for a fixed agenda, strict preferences over alternatives and satisfying some additional restrictions on the number of agents and alternatives. In our article, we also compare for the fixed agenda case (see Subsection 5.2) the consequences of Condorcet consistency with another condition that also involves comparing multiple preference profiles: pairwise justifiability, instead of strategy-proofness. However, our results apply to more general settings than theirs: agendas can vary, any number of agents and alternatives are allowed, and individual indifferences are considered.

The analysis of variable agendas has been considered in the literature starting from Arrow (1963), Fishburn (1973), Sen (1977), and Le Breton and Weymark (2011), among others. Allowing for variable agendas and considering collective choice functions on them can be used to analyze many interesting real-life circumstances in which specific subsets of alternatives are faced by society, and others are not.[2] Allowing for indifferences is not only natural but it also reinforces the coverage of our results by enlarging in a great measure the size of the preference domains where they apply.

An interesting and recent paper by Horan et al. (2019) also focuses, as we do, on variable agendas and it is the paper closest to our concerns, because its purpose is to extend May's characterization result for the majority rule with more than two alternatives and hence directly addresses the characteristics of collective choice rules in circumstances where a majority winner, if it exists, will definitely be chosen. One of their results is comparable to ours (see details in Section 3). Though relying on different properties, they show that, with three or more alternatives and strict preferences, only the rule that assigns to every problem under their consideration its strong Condorcet winner satisfies the conditions of anonymity, neutrality, positive responsiveness, and one à la Nash version of independence of irrelevant alternatives.

The article proceeds as follows: In Section 2, we provide notation and definitions. Section 3 investigates the connections between the requirement of Condorcet consistency and that of pairwise justifiability for collective choice functions under a condition on preference profiles (closed under improvements) and for which we present some examples in Section 4. Finally, Section 5 is devoted to the comparison between pairwise justifiability and other well-known properties in the two particular settings of a fixed profile and a fixed agenda, respectively. The proof of our results and some comments are gathered in the Appendix.

2. NOTATION AND DEFINITIONS

Let $N = \{1, 2, \ldots, n\}$ be a finite set of *agents* with $n \geqslant 2$. Let A be a finite set of *alternatives* with $\#A \geq 2$. We denote subsets of alternatives as B, B',... and we call them *agendas*. We denote by $\mathcal{A}$ the set of all nonempty subsets of A and by $\mathcal{B} \subseteq \mathcal{A}$ a *collection of subsets of alternatives*, or equivalently, *a collection of agendas*.

Let $\mathcal{R}$ be the set of all *preferences* on A (i.e., all complete, reflexive, and transitive binary relations on A). Elements of $\mathcal{R}$ are denoted by R_i, R_j,... The top of a preference $R_i \in \mathcal{R}$ in A, denoted by $t(R_i)$, is the set of alternatives $x \in A$ such that xP_iy for all $y \in A \setminus t(R_i)$. As usual, P_i and I_i denote the strict and indifference preference relation induced by R_i, respectively.

[2] In participatory budgeting, for instance, citizens are involved in the process of deciding how to spend part of a public budget and the set of feasible agendas consists of those collections of projects whose total cost is within the budget constraint. Another example is the composition of parliamentary committees that must satisfy a principle of proportionality, but the constraints imposed by this principle may vary across committees depending on their jurisdictions; or school admission problems with quotas, when the quotas may vary according to the admission criteria.

A *preference profile* is denoted by $R = (R_1, R_2, \ldots, R_n)$ and defines one preference for each agent $i \in N$. Let $\mathcal{R}^n$ be the set of all possible preference profiles, also called the *universal preference domain*, and let $\mathcal{D} \subseteq \mathcal{R}^n$ be a subset of preference profiles.

A *situation* is a pair consisting of a preference profile and an agenda: $(R, B) \in \mathcal{D} \times \mathcal{B}$.

A *collective choice function* on $\mathcal{D} \times \mathcal{B}$ is a mapping $C : \mathcal{D} \times \mathcal{B} \to A$ that assigns an alternative $C(R, B) \in B$ to each situation $(R, B) \in \mathcal{D} \times \mathcal{B}$. Along the paper, when defining collective choice functions, we impose and use that $C(R, \{x\}) = x$ for all $x \in A$.

Remark 1. Stricto sensu, the domain of the collective choice functions is the set of all admissible situations, that is, pairs formed by a preference profile and an agenda, on which the rule is defined. However, since along the paper we are interested in analyzing different sets of preference profiles, we shall use the term “preference domain” when referring to them.

We first define two properties on collective choice functions called anonymity and neutrality, each one capturing some idea of symmetry among agents and alternatives, respectively, as in Horan et al. (2019).[3]

Definition 1. A *collective choice function* C on $\mathcal{D} \times \mathcal{B}$ is anonymous on $\mathcal{D}' \times \mathcal{B}$ where $\mathcal{D}' \subseteq \mathcal{D}$ if, for any preference profile $R = (R_1, R_2, \ldots, R_n) \in \mathcal{D}'$, any $B \in \mathcal{B}$, and any permutation ρ of N such that $R_\rho \equiv (R_{\rho(1)}, R_{\rho(2)}, \ldots, R_{\rho(n)}) \in \mathcal{D}'$, then $C(R, B) = C(R_\rho, B)$.

To define neutrality for variable agendas, we have to specify how a permutation of the set of alternatives A is restricted to feasible agendas.

A permutation μ of A consists in a partition $\mathbb{P}$ of A such that for each subset $S_i \in \mathbb{P}$ with $S_i = \{a_1, \ldots, a_n\}$, there exists a sequence such that $\mu(a_i) = a_{i+1}$ for all $i = 1, \ldots, n-1$ and $\mu(a_n) = a_1$. For any $B \in \mathcal{B}$, define the restriction of μ to B as follows: for any $S_i \in \mathbb{P}$, if $S_i \subseteq B$, then $\mu_{|B}(a) = \mu(a)$ for all $a \in S_i$; if $S_i \cap B \neq \emptyset$ and $S_i \nsubseteq B$, then $\mu_{|B}(a) = a$ for all $a \in S_i$.

Definition 2. A *collective choice function* C on $\mathcal{D} \times \mathcal{B}$ is neutral on $\mathcal{D}' \times \mathcal{B}$ where $\mathcal{D}' \subseteq \mathcal{D}$ if, for any preference profile $R = (R_1, R_2, \ldots, R_n) \in \mathcal{D}'$, any $B \in \mathcal{B}$, and any permutation μ of A such that $\mu_{|B}(R) \equiv (\mu_{|B}(R_1), \mu_{|B}(R_2), \ldots, \mu_{|B}(R_n)) \in \mathcal{D}'$, then $\mu_{|B}(C(R, B)) = C(\mu_{|B}(R), B)$.

For an illustration of anonymity and neutrality, see Example 7 in the Appendix.

We now formalize *pairwise justifiability* for collective choice functions.[4]

Definition 3. A *collective choice function* C on $\mathcal{D} \times \mathcal{A}$ satisfies pairwise justifiability on $\mathcal{D}' \times \mathcal{B}$, $\mathcal{D}' \subseteq \mathcal{D}$ if, for any two situations (R, B), $(R', B') \in \mathcal{D}' \times \mathcal{B}$ such that $C(R, B) = x$, $C(R', B') = y$, and $x, y \in B \cap B'$, then there is some agent $i \in N$ and some alternative $z \in A \backslash \{x\}$ such that xP_iz and zR'_ix or xI_iz and zP'_ix.

This condition captures the basic motivational idea that was already provided in the introduction. To finish this section, we present three examples of collective choice functions either satisfying or violating pairwise justifiability. Example 1 defines a rule satisfying pairwise justifiability.

Example 1. Let $N = \{1, 2\}$, $A = \{a, b\}$, $\mathcal{B} = \{A\}$, and $\mathcal{D} = \mathcal{R}^n$ and let R^a be such that aPb, R^b such that bPa, and R^{ab} such that aIb. Let the collective choice function C be such that,

$$C(R, A) = \begin{cases} b & \text{if } b \in t(R_i) \forall i = \{1, 2\} \ \& \ t(R_i) \text{ is unique for some i} \in \{1, 2\} \\ a & \text{otherwise.} \end{cases}$$

[3] For fixed-agenda settings, Campbell and Kelly (2015) use other versions of both properties that are stronger to ours when comparable.

[4] See Barberà et al. (2024) for a version of pairwise justifiability for collective choice correspondences.

We prove that C satisfies pairwise justifiability for two cases, and leave the similar analysis of the remaining ones to the reader. Case (1) Take $R = (R^a, R^b)$ and $R' = (R^{ab}, R^b)$ where $C(R, A) = a$ and $C(R', A) = b$. Note that alternative a is strictly preferred to b under R_1^a, whereas a is indifferent to b under R_1^{ab}. Thus, $i = 1$ and $z = b$ in the definition of pairwise justifiability (this also shows that C violates Maskin monotonicity). Case (2) Take $R = (R^a, R^b)$ and $R' = (R^b, R^b)$ where $C(R, A) = a$ and $C(R', A) = b$. Note that alternative a is strictly preferred to b under R_1^a, whereas b is strictly preferred to a under R_1^b. Thus, $i = 1$ and $z = b$ in the definition of pairwise justifiability.

Examples 2 and 3 show collective choice functions violating pairwise justifiability, one in a situation with fixed profile and the other in a situation with fixed agenda, respectively.[5]

Example 2. Let $N = \{1, 2, 3\}$, $A = \{x, y, z\}$, $\mathcal{D} = \{R\}$ where R is such that xP_1yP_1z, yP_2zP_2x, zP_3xP_3y. The set of agendas is $\mathcal{B} = \{\{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$. Define the collective choice function C such that $C(R, \cdot)$ selects x at $B = \{x, y\}$, y at $B' = \{y, z\}$, z at $B'' = \{x, z\}$, and x at $B''' = \{x, y, z\}$. For (fixed-profile) pairwise justifiability to hold, since $C(R, B''), C(R, B''') \in B'' \cap B'''$, the rule should select the same at B'' and B''', but this is not the case.

Example 3. Let $n = 2$, $\mathcal{B} = \{A\}$ where $A = \{x, y, z\}$, and $\mathcal{D} = \mathcal{R}^n$. Let C be such that agent 1 is a dictator defined as follows: If $t(R_1)$ is not a singleton and $y \in t(R_1)$, then $C(R, A) = y$. If $t(R_1)$ is not a singleton and $y \notin t(R_1)$, that is, $t(R_1) = \{x, z\}$, then $C(R, A) = x$ if $y \notin t(R_2)$, and $C(R, A) = z$ otherwise. Let R, R' be such that $R_1 = R_1'$, $t(R_1) = \{x, z\}$, $t(R_2) = \{x, y\}$, $t(R_2') = \{z\}$. Then, $C(R, A) = z$ and $C(R', A) = x$ and pairwise justifiability is violated.

3. PAIRWISE JUSTIFIABILITY AND CONDORCET CONSISTENCY

In this section, we show that pairwise justifiability is strongly related to Condorcet consistency. We begin by formally defining the notion of Condorcet consistency in the context of collective choice functions and briefly elaborate upon its intrinsic interest.

For any preference profile $R \in \mathcal{D}$ and for any $B \in \mathcal{A}$, an alternative $y \in B$ *defeats* alternative z in B by *majority* at $R \in \mathcal{D}$ if the number of agents who strictly prefer y over z is strictly greater than the number of those who strictly prefer z over y.

We say that an alternative $y \in B \subseteq A$ is the (unique) *strong Condorcet winner at* (R, B) if y defeats any other alternative in B by majority at R. Since we consider collective choice functions, the consistency criterion has bite for all those situations for which, given the preference profile R, a strong Condorcet winner exists for the agenda B. Hence, our definition:

Definition 4. A *collective choice function* C on $\mathcal{D} \times \mathcal{A}$ is Condorcet consistent on $\mathcal{D}' \times \mathcal{B}$ where $\mathcal{D}' \subseteq \mathcal{D}$ if for each situation $(R, B) \in \mathcal{D}' \times \mathcal{B}$, we have that $C(R, B)$ selects the strong Condorcet winner at (R, B) when it exists.

For centuries now, this requirement, which demands that if one alternative is a strict majority winner over all others, it should be selected, has attracted much attention. This is understandable, because a first and foremost question in the theory of voting has been how to extend the notion of majority to the case where society faces more than two alternatives, especially as part of a criticism to the use of plurality voting, which can grossly deviate from any reasonable idea of respect to majorities. Although many social choice theorists find Condorcet's principle very attractive and compelling, others defend the use of scoring methods, which are notoriously distanced from this view.[6] But since respect of the Condorcet principle

[5] See Definitions 12 and 13 in Section 5 for formal details.

[6] See Chapter 9 in Moulin (1988) for an iluminating discussion of the tension between scoring methods and Condorcet consistent rules.

remains high in the list of favorite extensions of majority, we consider significant to exhibit its strong connection with pairwise justifiability, thus proving its interest and strength.

We now define the Condorcet preference domain $\mathcal{D}_{C\mathcal{B}}$ of $\mathcal{B}$ in $\mathcal{D}$.

Definition 5. Given a set $\mathcal{B}$ of agendas and a set $\mathcal{D}$ of admissible profiles, the Condorcet preference domain $\mathcal{D}_{C\mathcal{B}}$ of $\mathcal{B}$ in $\mathcal{D}$ is the subset of all preference profiles in $\mathcal{D}$ such that for all situations in $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$, there exists a strong Condorcet winner.

Whether a preference domain is Condorcet or not depends on the sets $\mathcal{B}$ and $\mathcal{D}$. For instance, given $\mathcal{B}$, adding preference profiles to the preference domain $\mathcal{D}$ may possibly enlarge the Condorcet preference domain $\mathcal{D}_{C\mathcal{B}}$. On the contrary, given $\mathcal{D}$, adding agendas to the collection $\mathcal{B}$ may possibly shrink the Condorcet preference domain $\mathcal{D}_{C\mathcal{B}}$. Consider Example 2 above, $\mathcal{D}_{C\mathcal{B}} = \emptyset$. However, $\mathcal{D}_{C\widetilde{\mathcal{B}}} = \mathcal{D} = \{R\}$ for the collection of agendas $\widetilde{\mathcal{B}} = \{B, B', B''\}$.

In Example 4, we provide an idea of the structure of a Condorcet preference domain. In Example 5, we insist on the relevance of the set of agendas $\mathcal{B}$.

Example 4. Let $N = \{1, 2, 3, 4\}$, $A = \{x, y, t\}$, $\mathcal{D} = \mathcal{R}^n$, $\mathcal{B} = \{B\}$ where $B = \{a, b\}$ for some pair $a, b \in A$ and denote as $\mathcal{R}_b^a$, $\mathcal{R}_a^b$, and $\mathcal{R}^{ab}$ the sets of all preference relations in $\mathcal{R}$ such that aPb, bPa, and aIb, respectively. Then, there does not exist a strong Condorcet winner at (R, B) for any preference profile R such that either (i) two agents have preference in $\mathcal{R}_b^a$ and two agents in $\mathcal{R}_a^b$, or (ii) one agent has preference in $\mathcal{R}_b^a$, one agent in $\mathcal{R}_a^b$, and two agents in $\mathcal{R}^{ab}$, or (iii) four agents have preferences in $\mathcal{R}^{ab}$. Therefore, $R \notin \mathcal{D}_{C\mathcal{B}}$, whereas for any other preference profile $R' \in \mathcal{D}$, $R' \in \mathcal{D}_{C\mathcal{B}}$.

Example 5. Consider three agents, four alternatives $A = \{x, y, z, t\}$, $\mathcal{B} = \{A\}$, and $\mathcal{D} = \mathcal{R}^n$. Then, any profile of preferences where each agent has x as the top alternative in $\mathcal{B}$ belongs to the Condorcet preference domain of $\mathcal{B}$ in $\mathcal{D}$. However, not all of these profiles can guarantee the existence of a strong Condorcet winner for $\mathcal{B} = \{A, B\}$ with $B = \{y, z, t\}$ in $\mathcal{D}$: consider the profile R such that $xP_1yP_1zP_1t$, $xP_2zP_2tP_2y$, $xP_3tP_3yP_3z$.

We now demonstrate that our proposed principle of justifiability, when applied to anonymous and neutral collective choice functions, is generally weaker than Condorcet consistency (see Proposition 1). However, it becomes equivalent to it (see Theorem 1) in well-known preference domains that satisfy a condition that we present immediately below.

As mentioned, the equivalence that we establish between pairwise justifiability and Condorcet consistency is preference domain-dependent. Yet, we will prove that it holds for some of the preference domains that are among the most popular in the analysis of collective choice functions and their properties. In principle, assessing the validity of statements regarding that equivalence for a given preference domain would require one proof for each separate case. A methodological contribution of this article is to identify a property of preference domains, that we call closedness under improvements, which is sufficient to prove that equivalence holds.

For any $x \in A$ and $R_i \in \mathcal{D}$, denote the *lower contour set* of R_i at x as $L(R_i, x) = \{y \in A : xR_iy\}$ and the strict counterpart as $\overline{L}(R_i, x) = \{y \in A : xP_iy\}$.

Definition 6. A preference domain $\mathcal{D} \subseteq \mathcal{R}^n$ satisfies closedness under improvements if for any $R \in \mathcal{D}$ and any $x, y \in A$ such that y defeats x by majority at R, there exists $R' \in \mathcal{D}$ satisfying the following requirements:

(1) for all $i \in N$, $L(R_i, x) \subseteq L(R_i', x)$, $\overline{L}(R_i, x) \subseteq \overline{L}(R_i', x)$, and $xR_i'y \iff xR_iy$, and

(2) for some permutation μ of A such that $x = \mu(y)$, $y = \mu(x)$, $\mu(R') \in \mathcal{D}$, and for a permutation ρ of N such that $\widehat{R} = \mu(R')_\rho \in \mathcal{D}$ and for all $i \in N$, $L(\widehat{R}_i, y) \subseteq L(R_i', y)$, $\overline{L}(\widehat{R}_i, y) \subseteq \overline{L}(R_i', y)$.

Closedness under improvements is a richness condition regarding preference domains of preference profiles. Starting from a preference profile R and two alternatives x and y, with x defeating y, there exists another admissible preference profile R' that maintains the order between x and y, ensures that x does not worsen with respect to any other alternative, and allows specific permutations of the names of the alternatives and individuals. The permutation of the names of the alternatives involves swapping the names x and y, and the subsequent permutation of the names of the agents results in a profile $\hat{R}$ where y does not worsen with respect to any other alternative compared to R'. The condition is clearly satisfied by the universal and the strict universal preference domain, but also by the order restricted, the single-peaked, and the extended single-peaked preference domains.[7] To gain intuition about what this condition requires, consider the following example:

EXAMPLE 6. Let $A = \{x, m, y\}$, $x < m < y$, $N = \{1, 2, 3, 4, 5\}$ and suppose that agents have single-peaked preferences (see Definition 10). Consider a single-peaked preference profile R such that agents 1 and 2 have their peak on x, agents 3 and 4 have their peak on m and x is their worst alternative, and agent 5 has their peak on y. Take alternatives y, x where y defeats x by majority at R. Closedness under improvements requires the existence of a single-peaked preference profile R' satisfying parts 1 and 2 in Definition 6. Consider R' such that $R'_i = R_i$ for all agent $i \neq 4$ and $R'_4 : yP'_4mP'_4x$. Note that:

1. for every agent i, the relative position of x and y is the same in R and in R', $L(R'_i, x) = L(R_i, x)$ and $\overline{L}(R'_i, x) = \overline{L}(R_i, x)$;
2. there exists a single-peaked preference profile R'' that is a permutation of the profile R', where the alternatives x and y have switched, $\mu(R') = R''$ (i.e., in R'', agents 1 and 2 have their peak on y, agents 4 and 5 have their peak on x, and agent 3 has the peak on m and has y as the worst alternative); and finally,
   there exists a single-peaked preference profile $\hat{R}$ that is obtained as a permutation ρ of N starting from $R'' = \mu(R')$ such that $L(\widehat{R}_i, y) \subseteq L(R'_i, y)$, $\overline{L}(\widehat{R}_i, y) \subseteq \overline{L}(R'_i, y)$ (i.e., $\hat{R} = \mu(R')_\rho$, where agents 1 and 4 exchange preference orderings, and similarly, agents 2 and 5 exchange preference orderings).

Example 7 in the Appendix presents a preference domain violating closedness under improvements.

Our Theorem 1 offers a characterization of Condorcet consistent rules defined on any closed under improvements preference domain where a strong Condorcet winner exists.

THEOREM 1. *Let $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{D} \subseteq \mathcal{R}^n$ such that $\mathcal{D}_{C\mathcal{B}}$ is closed under improvements, and C be a collective choice function C on $\mathcal{D} \times \mathcal{B}$. C satisfies anonymity, neutrality, and pairwise justifiability on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$ if and only if C is Condorcet consistent on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$.*

The proof of Theorem 1 is in the Appendix. The proof in the "if" direction is straightforward. The "only if" part is proved by contradiction and we provide here an intuition based on Example 6 above in the single-peaked preference domain for the case $\mathcal{B} = \{A\}$. Suppose that C is not Condorcet consistent and $C(R, A) = x$. By pairwise justifiability, $C(R', A) = x$ since for no agent, any alternative has improved with respect to x when going from R to R'. By neutrality, $C(R'', A) = y$. By anonymity, $C(\hat{R}, A) = y$. However, $\hat{R} = R$ and therefore we have reached the desired contradiction.

Here are a few remarks on Theorem 1.

REMARK 2. For all the Condorcet preference domains $\mathcal{D}_{C\mathcal{B}}$ satisfying Definition 6, the equivalence result holds. As shown in Propositions 2 and 3 in Section 4, this is the case, for

[7] See Section 4 for the definition and the analysis of these preference domains concerning our richness preference domain condition.

example, when the preference domain is either the universal preference domain, the strict universal preference domain, the preference domain of single-peakedness, order-restricted preference domains, or the extended single-peaked preference domain (see precise definitions of these conditions in the following section). However, as Example 7 in the Appendix and the one in Footnote 7 show, the equivalence may not hold for $\mathcal{D}_{C\mathcal{B}}$ violating closedness under improvements.

REMARK 3. The result of Theorem 1 can be extended, in one direction, to show that any Condorcet consistent collective choice function is anonymous, neutral, and satisfies pairwise justifiability on any preference subdomain of a Condorcet preference domain, as stated in the following proposition proved in the Appendix.

PROPOSITION 1. *Let $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{D} \subseteq \mathcal{R}^n$, and C be a collective choice function on $\mathcal{D} \times \mathcal{B}$. If C is Condorcet consistent on $\mathcal{D}' \times \mathcal{B}$, then C satisfies pairwise justifiability on $\mathcal{D}' \times \mathcal{B}$ for any $\mathcal{D}' \subseteq \mathcal{D}_{C\mathcal{B}}$.*

REMARK 4. Condorcet consistency is not always a consequence of pairwise justifiability. In particular, this may be false for all preference subdomains of the Condorcet preference domain of $\mathcal{B}$ in $\mathcal{D}$.[8]

REMARK 5. The result in Proposition 1 does not hold if $\mathcal{D}' \nsubseteq \mathcal{D}_{C\mathcal{B}}$. See Example 2 where the collective choice function C satisfies Condorcet consistency on $\mathcal{D}' \times \mathcal{B}$ but it violates pairwise justifiability on $\mathcal{D}' \times \mathcal{B}$ where $\mathcal{D}' = \{R\} \nsubseteq \mathcal{D}_{C\mathcal{B}} = \emptyset$.

It is important to note that, although Condorcet consistency is usually predicated for situations involving a single agenda, we are also able to cover the consequences of pairwise justifiability on the choice of strong Condorcet winners by collective choice rules defined over situations that include multiple agendas. For example, it may be used in the analysis of the many cases in political economy where the preference profiles satisfy intermediateness and society may confront agendas of different size. In these cases, for any agenda, the alternative preferred by the median voter is a strong Condorcet winner.

Finally, we add more precise comments regarding some significant differences between the results in Horan et al. (2019) and ours, both in terms of their respective purposes and regarding more technical requirements. In our case, the main purpose is to connect two conditions, Condorcet consistency and pairwise justifiability, which are formally quite different, and to show that they are, in fact, equivalent in many circumstances. That leads us to concentrate on functions defined over domains that differ from those on which the functions considered by Horan et al. (2019) are. Specifically, their domains are restricted to only include situations (preference profiles-agendas) where a Condorcet winner exists, whereas we do not exclude the possibility of cyclical patterns to arise in admissible situations. This is justified because, to establish precise relations between the two conditions we focus on, we must consider the performance of each one of them, and pairwise justifiability is a well-defined restriction even in those cases where Condorcet consistency does not have bite. Of course, we later restrict attention in our statements to what we call Condorcet preference domains, but the domains considered in each paper are quite different: in particular, our domains are the Cartesian product of subsets of profiles of preferences, on the one hand, and subsets of agendas on the other hand, and theirs are not. Other formal differences arise because, whereas they use two properties on rules, one of them applying to situations with a fixed profile and a variable agenda, the other

[8] Let $N = \{1, 2\}$, $A = \{x, y, z\}$, $\mathcal{B} = \{A\}$, and R, R' be two admissible profiles such that yP_1xP_1z, yP_2xP_2z and $yP'_1zP'_1x$, $yP'_2zP'_2x$. Define C such that $C(R, B) = x$ and $C(R', B) = z$. Note that C satisfies pairwise justifiability but it does not choose the strong Condorcet winner at any feasible situation. Note that $\mathcal{D} = \{R, R'\}$ violates closedness under improvements.

applying to changing profiles with the same agenda, our notion of pairwise justifiability combines these two conditional situations with others where both the agenda and the profile may vary. Consequently, we do not need to impose the condition of independence that is part of their definition of collective choice rules. Moreover, it is also important to notice that our formulation allows for agents to be indifferent among alternatives whereas, in their Theorem 1, they consider strict preferences.

4. PREFERENCE DOMAINS SATISFYING CLOSEDNESS UNDER IMPROVEMENTS

In this section, we consider several families of preference profiles: the universal preference domain $\mathcal{R}^n$, the strict universal preference domain $\mathcal{P}^n \subseteq \mathcal{R}^n$ that is the subset of all preference profiles where agents' preferences on A are *strict* (i.e., antisymmetric), and the order restricted, the single-peaked, and the extended single-peaked preference domains that we define below.[9] We show that each one of these preference domains and its corresponding Condorcet preference domain of $\mathcal{B}$ satisfies closedness under improvements.

To define order restriction (or intermediateness), we need some additional notation. For any $B, C \subseteq N$, $B \cap C = \emptyset$ and any order $\succ$ of N, if for any agent $i \in B$ and any $j \in C$, $i \succ j$, we say that $B \succ C$.

DEFINITION 7. A preference profile R is order restricted on A if there is an order $\succ_R$ of N such that for all distinct $x, y \in A$,
$\{i \in N : xP_iy\} \succ_R \{i \in N : xI_iy\} \succ_R \{i \in N : yP_ix\}$ or
$\{i \in N : yP_ix\} \succ_R \{i \in N : xI_iy\} \succ_R \{i \in N : xP_iy\}$.

DEFINITION 8. A preference domain $\mathcal{D}^{\mathcal{OR}}$ is order restricted on A if any preference profile $R \in \mathcal{D}^{\mathcal{OR}}$ is order restricted.

Next, we introduce two types of preference domains based on the notion of single-peakedness.

DEFINITION 9. A preference profile R is single-peaked on A if there exists a linear order $>$ of the set of alternatives such that for any individual $i \in N$, R_i is single-peaked on A relative to the $>$, that is,
(1) R_i has a unique maximal element $t(R_i)$, called the top of i, and
(2) for all $y, z \in A$, $[z < y \leq t(R_i)$ or $z > y \geq t(R_i)] \rightarrow yP_iz$.

DEFINITION 10. Let $\mathcal{D}^{\mathcal{S},>}$ be the set of all preference profiles that are single-peaked with respect to the same order $>$. Each one of these sets is called a single-peaked preference domain.

We now define the larger preference domain of extended single-peaked profiles.

DEFINITION 11. Let $\mathcal{D}^{\mathcal{E}}$ be the set of all preference profiles that are single-peaked relative to some order. This set is the extended single-peaked preference domain.

In Propositions 2 and 3 below, we prove that the Condorcet preference domains of $\mathcal{B}$ in different preference domains of interest are closed under improvements. The same proof procedure applies to establish that these preference domains satisfy the condition. We give a common statement that covers the universal, strict universal, order restricted, extended single-peaked, and we state the same conclusion for the case of single-peaked preference domains. This is because, although the use of these propositions allows us to show that our equivalence

[9] See also Grandmont (1978) and Rothstein (1990) for formal definition of order restriction, Penn et al. (2011) for the definition of extended single-peakedness, and Black (1958) for the definition of single-peakedness.

result holds for all of them, the proof for the case of single-peakedness is different from that of the other preference domains. In fact, our techniques are similar to those used in other papers that investigate the nature of Condorcet consistency, and in particular to those employed to extend May (1952)'s result[10] to any number of alternatives (Horan et al., 2019). The choice of our differentiated line of reasoning, resorting to a general property and then proving that it is satisfied under different circumstances, allows us to unify the consequences of our properties on collective choice functions and we think that it justifies our specific course of action.

Proposition 2. *Let $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{D} \in \{\mathcal{R}^n, \mathcal{P}^n, \mathcal{D}^{\mathcal{E}}, \mathcal{D}^{\mathcal{OR}}\}$. The Condorcet preference domain $\mathcal{D}_{\mathcal{CB}}$ of $\mathcal{B}$ in $\mathcal{D}$ satisfies closedness under improvements.*

We now analyze the case of single-peaked preference domains relative to a given order.

Proposition 3. *Let $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{D} = \mathcal{D}^{S,>}$. The Condorcet preference domain $\mathcal{D}_{\mathcal{CB}}$ of $\mathcal{B}$ in $\mathcal{D}$ satisfies closedness under improvements.*

The strategy of the proof is the same in Propositions 2 and 3: starting from a profile R and any pair of alternatives y, x such that y defeats alternative x, we must construct a preference profile R' satisfying parts 1 and 2 in Definition 6. The preference profile R' such that for each agent, y and x are ranked as in the original profile R, and for all agents, they are the top-two alternatives and all the other alternatives are ordered in the same way works in the case of Proposition 2. However, the profile obtained using this procedure may not belong to the single-peaked preference domain. Therefore, we have to adapt the proof to Proposition 3. The construction of R' heavily depends on both the starting profile R and the set of alternatives that are between x and y as can be seen in the proof. The procedure used in Proposition 2 would only work when there is no alternative between x and y.

5. FIXED PROFILE WITH VARIABLE AGENDAS AND FIXED AGENDA WITH VARIABLE PROFILES

Until now, to state our results, we have used the condition of pairwise justifiability in its full force, applying it in the general case where both the preference profile and the agenda may vary. This may obscure the connection between our condition and other important ones in the literature that only consider changes in one of the two aspects of our situations, by either fixing the preference profile or the agenda. In what follows we discuss these two polar cases and, in particular, use them to compare the property of pairwise justifiability with others.

5.1. *Fixed Profile with Variable Agendas.* Let $R \in \mathcal{R}^n$ be a preference profile. A collective choice function on $\mathcal{A}$ is a mapping $C : \mathcal{A} \to A$ that assigns an alternative $C(B) \in B$ for each agenda $B \in \mathcal{A}$.

Note that pairwise justifiability is a consistency property.

Definition 12. A collective choice function C on $\mathcal{A}$ satisfies pairwise justifiability on $\mathcal{A}$ if for any $B, B' \in \mathcal{A}$ such that $B \cap B' \neq \varnothing$, if $C(B), C(B') \in B \cap B'$, then $C(B) = C(B')$.

Observe that, focusing on the case $\mathcal{B} = \mathcal{A}$, Sen's α-consistency is equivalent to fixed-profile pairwise justifiability.[11] The proof that Sen's α-consistency implies fixed-profile pairwise justifiability is straightforward noticing that for any pair of nonempty intersection agendas B and B', one has to apply Sen's α-consistency twice with $S = B \cap B'$ and $T^1 = B$ and $T^2 = B'$. To show the converse, let B, B' such that $B \subset B'$ thus $B \cap B' = B$ and apply fixed-profile pairwise justifiability.

[10] We appreciate the contribution of an anonymous referee who raised this point.

[11] When any possible agenda is feasible, Sen's α-consistency requires that if the choice in a larger agenda belongs to a smaller agenda, then the choice in the smaller one must be the same. See the formal definition in Sen (1977).

Note that in this framework, anonymity and neutrality are vacuously satisfied since $\mathcal{D}' = \mathcal{D} = \{R\}$, and thus, the identity is the unique permutation of agents and alternatives, respectively, such that $\rho(R), \mu(R) \in \mathcal{D}'$. Moreover, note that $\mathcal{D}'$ does not satisfy closedness under improvements because there is only one admissible preference profile and it violates part 2 of the condition. Thus, Theorem 1 does not apply. In fact, a dictatorial rule with fixed profile satisfies pairwise justifiability but it may violate Condorcet consistency.

5.2. *Fixed Agenda with Variable Profiles.* Let A be the fixed agenda. A collective choice function on $\mathcal{D} \subseteq \mathcal{R}^n$ is a mapping $C : \mathcal{D} \to A$ that assigns an alternative $C(R) \in A$ for each preference profile $R \in \mathcal{D}$.

Definition 13. A *collective choice function C* on $\mathcal{D}$ satisfies pairwise justifiability on $\mathcal{D}'$, $\mathcal{D}' \subseteq \mathcal{D}$ if, for any two preference profiles R, $R' \in \mathcal{D}'$ such that $C(R) = x$, $C(R') = y$, then there is some agent $i \in N$ and some alternative $z \in A \backslash \{x\}$ such that xP_iz and zR'_ix or xI_iz and zP'_ix.

Note that in this framework with the fixed agenda A, we obtain the following result as a corollary of Theorem 1 when applied for $\mathcal{B} = \{A\}$.

Corollary 1. *Let $\mathcal{D} \subseteq \mathcal{R}^n$ be such that $\mathcal{D}_{CA}$ is closed under improvements, and C be a collective choice function C on $\mathcal{D}$. C satisfies anonymity, neutrality, and pairwise justifiability on $\mathcal{D}_{CA}$ if and only if C is Condorcet consistent on $\mathcal{D}_{CA}$.*

In this setting where the agenda is fixed, we can compare pairwise justifiability with other well-known properties like strategy-proofness, Maskin monotonicity, and strong positive association, among others. When preferences are strict, Maskin monotonicity and pairwise justifiability are equivalent (by definition). In their Footnote 3, Müller and Satterthwaite (1977) show that strategy-proofness and strong positive association (or Maskin monotonicity) are equivalent in the universal strict domain of preference profiles. However, they also mention that for restricted domains of strict preference profiles, strategy-proofness implies strong positive association, but the converse does not necessarily hold. If we admit indifferences, strategy-proofness and pairwise justifiability are independent (see Examples 8 and 9 in the Appendix) and, by definition, Maskin monotonicity implies pairwise justifiability but the converse does not hold. Example 10 in the Appendix shows that Condorcet consistency and Maskin monotonicity are independent of each other when indifferences are allowed. Other weakenings of Maskin monotonicity that resemble fixed-agenda pairwise justifiability have been considered in the literature. See, for example, Sanver (2006), Berga and Moreno (2009), and Barberà et al. (2012) among others.[12]

As mentioned in the introduction, Campbell and Kelly (2003, 2015) study the relation between Condorcet consistency and strategy-proofness. First, their results apply to strict preference domains, whereas our results also apply to more preference domains including domains that admit indifferences. Secondly, Campbell and Kelly (2003) impose an odd number of agents, and Campbell and Kelly (2015) consider a particular relationship between the number of agents and alternatives.

[12] Sanver (2006) introduces a property for a fixed agenda named almost monotonicity, which is equivalent to our property, and uses it in a problem of implementation by awards. Berga and Moreno (2009) study the provision of one public good when agents have single-plateaued preferences and define a property that in their setting coincides with pairwise justifiability. Also Barberà et al. (2012) define a similar property called monotonicity to identify preference domains where this property joint with an invariance property (named reshuffling invariance) is necessary and sufficient conditions for strategy-proofness.

APPENDIX A

EXAMPLE 7. Consider $N = \{1, 2\}$, $A = \{x, y, z\}$, $\mathcal{B} = \{B, B'\}$ where $B = \{x, y\}$, and $B' = \{x, z\}$. Let R be such xI_1yI_1z and yP_2xP_2z, $\hat{R}$ such that $x\hat{I}_1y\hat{I}_1z$ and $y\hat{P}_2z\hat{P}_2x$ and $\mathcal{D} = \{R, \hat{R}\}$.

This example serves three purposes: (i) to illustrate the process of determining which collective choice functions are anonymous and neutral; (ii) to demonstrate the method for determining whether a preference domain satisfies closedness under improvements; (iii) to illustrate that if the Condorcet preference domain $\mathcal{D}_{C\mathcal{B}}$ violates closedness under improvements, it is possible to construct an anonymous, neutral, and pairwise justifiable collective choice function that violates Condorcet consistency.

(1) First, we discuss which collective choice functions are anonymous on $\mathcal{D} \times \mathcal{B}$. The unique permutation of agents that keeps the permuted profile in the preference domain is the identity. Thus, any collective choice function on $\mathcal{D} \times \mathcal{B}$ satisfies anonymity. Second, we check which collective choice functions are neutral on $\mathcal{D} \times \mathcal{B}$. Among the six possible permutations of A, the only one that does not keep the permuted profile in the preference domain is μ such that $S_1 = \{x, y\}$, $S_2 = \{z\}$. Consider, in fact, the situation (R, B): since $S_1 \subseteq B$, $\mu_{|B}(x) = y$, $\mu_{|B}(y) = x$, and $\mu_{|B}(z) = z$. Then, $\mu_{|B}(R) = R''$ where $yI_1''xI_1''z$ and $xP_2''yP_2''z$, which is not in the preference domain. Consider the following permutations of A, μ' such that $S_1' = \{y, z\}$, $S_2' = \{x\}$; μ'' such that $S'' = \{x, y, z\}$; and μ''' such that $S''' = \{x, z, y\}$. For any $\hat{B} \in \mathcal{B}$ and any nonsingleton $\hat{S}$, we have $\hat{S} \cap \hat{B} \neq \emptyset$ and $\hat{S} \nsubseteq \hat{B}$. Thus, for any situation and any $\bar{\mu} \in \{\mu', \mu'', \mu'''\}$, $\bar{\mu}_{|\hat{B}}(x) = x$, $\bar{\mu}_{|\hat{B}}(y) = y$, $\bar{\mu}_{|\hat{B}}(z) = z$, $\bar{\mu}_{|\hat{B}}(R) = R$, and $\bar{\mu}_{|\hat{B}}(\hat{R}) = \hat{R}$. Finally, consider $\tilde{\mu}$: $\tilde{S}_1 = \{x, z\}$, $\hat{S}_2 = \{y\}$. For B, $\tilde{\mu}_{|B}(x) = x$, $\tilde{\mu}_{|B}(y) = y$, $\tilde{\mu}_{|B}(z) = z$, $\tilde{\mu}_{|B}(R) = R$, $\tilde{\mu}_{|B}(\hat{R}) = \hat{R}$. For B', $\tilde{\mu}_{|B'}(x) = z$, $\tilde{\mu}_{|B'}(y) = y$, $\tilde{\mu}_{|B'}(z) = x$, $\tilde{\mu}_{|B'}(R) = \hat{R}$, and $\tilde{\mu}_{|B'}(\hat{R}) = R$. Permutation $\tilde{\mu}$ is the only one that imposes a constraint on collective choice functions to satisfy neutrality. In fact, a collective choice function satisfies neutrality if and only if $C(R, B') \neq C(\hat{R}, B')$.

(2) We verify whether the preference domain $\mathcal{D} = \{R, \hat{R}\}$ satisfies closedness under improvements. Consider the preference profile $R \in \mathcal{D}$ and alternatives y, x where y defeats x by majority at R. Observe that part 1 of Definition 6 is satisfied only if $R' = R$. However, part 2 is violated since the unique permutation of A exchanging the role of x and y is μ: $S_1 = \{x, y\}$, $S_2 = \{z\}$, and $\mu(R) \notin \mathcal{D}$. Therefore, the preference domain $\mathcal{D} = \{R, \hat{R}\}$ does not satisfy closedness under improvements.

We demonstrate the existence of an anonymous, neutral, and pairwise justifiable collective choice function that violates Condorcet consistency. Note that $\mathcal{D}_{C\mathcal{B}} = \mathcal{D}$ and, by (2) above, it violates closedness under improvements. Let C be such that $C(R, B) = C(\hat{R}, B) = C(R, B') = x$ and $C(\hat{R}, B') = z$. The proof that C satisfies anonymity and neutrality follows from the above discussion. We prove that C satisfies pairwise justifiability; Since $z \notin B$, we only have to look at two situations: (R, B') and $(\hat{R}, B')$. Since xP_2z and $z\hat{P}_2x$, then pairwise justifiability is satisfied. Notice that C violates Condorcet consistency because alternative y is the strong Condorcet winner at (R, B), and $C(R, B) = x$.

Theorem 1 Let $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{D} \subseteq \mathcal{R}^n$ such that $\mathcal{D}_{C\mathcal{B}}$ is closed under improvements, and C be a collective choice function C on $\mathcal{D} \times \mathcal{B}$. C satisfies anonymity, neutrality, and pairwise justifiability on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$ if and only if C is Condorcet consistent on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$.

Proof of Theorem 1. First, we prove the "if" implication. Since C is Condorcet consistent on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$, then C is anonymous and neutral on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$. Anonymity and neutrality are straightforward. We now prove that C satisfies pairwise justifiability on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$. Take any two situations (R, B), $(R', B') \in \mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$ such that $C(R, B) = x$, $C(R', B') = y$, and $x, y \in B \cap B'$. Since $C(R, B) = x$, the number of agents who strictly prefer x over y is greater than that of those who strictly prefer y over x at R. Also, since $C(R', B') = y$, the number of agents who strictly prefer y over x is greater than that of those who strictly prefer x over y at R'. Therefore, there exists some agent i such that either xP_iy and yR'_ix or xI_iy and yP'_ix. Thus, pairwise justifiability holds.

We now prove the "only if" statement by contradiction. Let C satisfy anonymity, neutrality, and pairwise justifiability on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$. Suppose that C is not Condorcet consistent on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$. Then, there exists a situation $(R, B) \in \mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$ such that y is the strong Condorcet winner at (R, B) and $C(R, B) = x \neq y$. Since $\mathcal{D}_{C\mathcal{B}}$ is closed under improvements, there exists R' satisfying parts 1 and 2. By part 1, there is no alternative $z \in A\backslash\{x\}$ and no agent $i \in N$ such that z has improved with respect to x. That is, by pairwise justifiability of C, $C(R', B) = x$. By part 2, there exists a permutation μ of A such that $\mu(x) = y$, $\mu(y) = x$, and $R'' = \mu(R') \in \mathcal{D}_{C\mathcal{B}}$. By neutrality, $C(R'', \mu(B)) = y$. Also by part 2, there exists a permutation ρ of N such that $\widehat{R} = \mu(R')_\rho \in \mathcal{D}_{C\mathcal{B}}$. By anonymity, $C(\widehat{R}, \mu(B)) = y$. Also by part 2, for any $i \in N$, $L(\widehat{R}_i, y) \subseteq L(R'_i, y)$, $\overline{L}(\widehat{R}_i, y) \subseteq \overline{L}(R'_i, y)$. Therefore, from $\widehat{R}$ to R', there is no alternative $z \in A\backslash\{y\}$ and no agent $i \in N$ such that z has improved with respect to y. Thus, by pairwise justifiability of C, $C(R', B) = C(\widehat{R}, \mu(B)) = y$. Thus, we get the desired contradiction. ■

Proposition 1 Let $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{D} \subseteq \mathcal{R}^n$, and C be a collective choice function on $\mathcal{D} \times \mathcal{B}$. If C is Condorcet consistent on $\mathcal{D}' \times \mathcal{B}$, then C satisfies pairwise justifiability on $\mathcal{D}' \times \mathcal{B}$ for any $\mathcal{D}' \subseteq \mathcal{D}_{C\mathcal{B}}$.

Proof of Proposition 1. Let C be Condorcet consistent on $\mathcal{D}' \times \mathcal{B}$ where $\mathcal{D}' \subseteq \mathcal{D}_{C\mathcal{B}}$. Take any two situations (R, B), $(R', B') \in \mathcal{D}' \times \mathcal{B}$ where $C(R, B) = x$, $C(R', B') = y$, $x, y \in B \cap B'$, Since $C(R, B) = x$, x defeats y in B by majority at R. Also since $C(R', B') = y$, y defeats x in B' at R'. Therefore, there exists some agent i such that xP_iy and yR'_ix. ■

Proposition 2 Let $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{D} \in \{\mathcal{R}^n, \mathcal{P}^n, \mathcal{D}^{\mathcal{E}}, \mathcal{D}^{\mathcal{OR}}\}$. The Condorcet preference domain $\mathcal{D}_{C\mathcal{B}}$ of $\mathcal{B}$ in $\mathcal{D}$ satisfies closedness under improvements.

Proof of Proposition 2. Let $\mathcal{B} \subseteq \mathcal{A}$ and $R \in \mathcal{D}_{C\mathcal{B}}$. Take any pair of alternatives $x, y \in A$ such that y defeats x by majority at R. Consider the partition of N such that $Y = \{i \in N : yP_ix\}$, $X = \{i \in N : xP_iy\}$, and $L = \{i \in N : xI_iy\} = N\backslash(Y \cup X)$. Note that the number of agents in Y is greater than that in X.

Let R' be the following preference profile: (i) for all $i \in X$, $t(R'_i) = x$ and yP'_iz for any $z \in A \setminus \{x, y\}$, (ii) for all $i \in Y$, $t(R'_i) = y$ and xP'_iz for any $z \in A \setminus \{x, y\}$, (iii) for all $i \in L$, $t(R'_i) = \{x, y\}$, yP'_iz and xP'_iz for any $z \in A \setminus \{x, y\}$, and (iv) the relative order of the alternatives in $A \setminus \{x, y\}$ is the same and is strict.

Note that trivially, $R' \in \mathcal{D}$ for $\mathcal{D} \in \{\mathcal{R}^n, \mathcal{P}^n\}$. For $\mathcal{D} = \mathcal{D}^{\mathcal{E}}$, let $>_{R'}$ be such that $x > y > z$ for any $z \in A \setminus \{x, y\}$ and the alternatives in $A \setminus \{x, y\}$ are ordered following the same and strict ordering of the preferences of the agents at R' as in (iv). For $\mathcal{D} = \mathcal{D}^{\mathcal{OR}}$, let $\succ_{R'} = \succ_R$. Note also that $R' \in \mathcal{D}_{C\mathcal{B}}$ because for any B such that $y \in B$, $CW(R', B) = y$. Also, for any B such that $y \notin B$ and $x \in B$, $CW(R', B) = x$. Otherwise, the unanimous most preferred alternative in B at R' is the strong Condorcet winner.

Now, we show that R' satisfies parts 1 and 2 of Definition 6.

Part 1 For any $i \in N$, $L(R_i, x) \subseteq L(R'_i, x)$ and $\overline{L}(R_i, x) \subseteq \overline{L}(R'_i, x)$ and $xR'_iy \iff xR_iy$.

Part 1 is satisfied by construction of R' because for all $i \in N$, for all $z \in A \setminus \{x, y\}$, xP'_iz, yP'_iz and the preferences of the agents over $\{x, y\}$ are the same in R and R'.

Part 2 There exists a permutation μ of A such that $x = \mu(y)$, $y = \mu(x)$, $\mu(R') \in \mathcal{D}_{C\mathcal{B}}$.

Define μ of A such that $z = \mu(z)$, for all $z \in A \setminus \{x, y\}$, $x = \mu(y)$, and $y = \mu(x)$.

Note that trivially, $\mu(R') \in \mathcal{D}$ for $\mathcal{D} \in \{\mathcal{R}^n, \mathcal{P}^n\}$. For $\mathcal{D} = \mathcal{D}^{\mathcal{E}}$, let $>_{\mu(R')} = >_{R'}$. For $\mathcal{D} = \mathcal{D}^{\mathcal{OR}}$, let $\succ_{\mu(R')} = \succ_{R'}$. Note that $\mu(R') \in \mathcal{D}_{C\mathcal{B}}$ because for any B such that $x \in B$, $CW(\mu(R'), B) = x$. Also, for any B such that $x \notin B$ and $y \in B$, $CW(\mu(R'), B) = y$. Otherwise, the unanimous most preferred alternative in B at $\mu(R')$ is the strong Condorcet winner.

Moreover, there exists a permutation ρ of N such that $\widehat{R} = \mu(R')_\rho \in \mathcal{D}_{C\mathcal{B}}$ and for any $i \in N$, $L(\widehat{R}_i, y) \subseteq L(R'_i, y)$, $\overline{L}(\widehat{R}_i, y) \subseteq \overline{L}(R'_i, y)$.

It is immediate to check that, for any $B \in \mathcal{B}$, we have that $CW(\widehat{R}, B) = CW(\mu(R'), B)$. Therefore, $\widehat{R} \in \mathcal{D}_{C\mathcal{B}}$.

By construction of R', y is the second best alternative in A for any $j \in Y$ at $\mu(R')$ and for any $i \in X$ at R'_i. Since $|X| < |Y|$, for $|X|$ agents $j \in Y$, there exists $i \in X$ such that $L(\mu(R'_j), y) = L(R'_i, y) = A \setminus \{x\}$ and $\overline{L}(\mu(R'_j), y) = \overline{L}(R'_i, y) = A \setminus \{x, y\}$. Symmetrically, by construction of R', y is the unique best alternative in A at R' for any $j \in Y$ and at $\mu(R'_i)$ for any $i \in X$. Since $|X| < |Y|$, for any $i \in X$, there exists $j \in Y$ such that $L(\mu(R'_i), y) = L(R'_j, y) = A$, $\overline{L}(\mu(R'_i), y) = \overline{L}(R'_j, y) = A \setminus \{y\}$.

By construction of R', for any $\ell \in L$, $L(\mu(R'_\ell), y) = L(R'_\ell, y) = A$, $\overline{L}(\mu(R'_\ell), y) = \overline{L}(R'_\ell, y) = A \setminus \{x, y\}$.

By construction of R', for any $|Y| - |X|$ agents left in Y, y is the second best alternative in A at $\mu(R')$ and $A \setminus \{x\} = L(\mu(R'_j), y) \subset L(R'_j, y) = A$, $A \setminus \{x, y\} = \overline{L}(\mu(R'_j), y) \subset A \setminus \{y\} = \overline{L}(R'_j, y)$.

We now define ρ of N as follows: (i) $j = \rho(i)$, for $|X|$ agents $j \in Y$ and any $i \in X$; (ii) $i = \rho(j)$, for $|X|$ agents $j \in Y$ and any $i \in X$, and (iii) $k = \rho(k)$, for any $k \in L$ and the remaining $|Y| - |X|$ agents in Y. ■

For any $x \in A$ and $R_i \in \mathcal{D}$, define the upper contour set as $U(R_i, x) = \{y \in A : yR_ix\}$ and the strict counterpart as $\overline{U}(R_i, x) = \{y \in A : yP_ix\}$.

Proposition 3 Let $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{D} = \mathcal{D}^{S,>}$. The Condorcet preference domain $\mathcal{D}_{C\mathcal{B}}$ of $\mathcal{B}$ in $\mathcal{D}$ satisfies closedness under improvements.

Proof of Proposition 3. Let $\mathcal{B} \subseteq \mathcal{A}$ and $R \in \mathcal{D}^{S,>}_{C\mathcal{B}}$. Take any pair of alternatives $x, y \in A$ such that y defeats x by majority at R. Suppose without loss of generality that $x < y$. Consider the partition of N such that $Y_1 = \{i \in N : yP_ix \ \& \ t(R_i) \geq y\}$, $Y_2 = \{i \in N : yP_ix \ \& \ t(R_i) < y\}$, $X_1 = \{i \in N : xP_iy \ \& \ t(R_i) \leq x\}$, $X_2 = \{i \in N : xP_iy \ \& \ t(R_i) > x\}$, and $L = \{i \in N : xI_iy\} = N \setminus (Y \cup X)$ where $Y \equiv Y_1 \cup Y_2$ and $X \equiv X_1 \cup X_2$,. Note that the number of agents in Y is greater than that in X. Let $M = \{z \in A : x \leq z \leq y\}$ and let M^c its complement set in A.

We have to distinguish two cases

Case (a): $|X| \leq |Y_1|$. Note that if $|X| = |Y_1|$, then Y_2 is not empty because $|Y| > |X|$. Let $J \subseteq Y_1$ with $|J| = |Y_1| - |X|$. Note that if $|X| < |Y_1|$, J is not empty. Therefore, $Y_2 \cup J$ is not empty. If X_2 is not empty, let $k \in X_2$ be one of the agents for which $U(R_k, x) \subseteq U(R_i, x)$ for any $i \in X_2$. Let R' be a preference profile in $\mathcal{D}^{S,>}$ defined as follows: (i) for all $i \in X_1$, $t(R'_i) = x$, for all $i \in Y_1 \setminus J$, $t(R'_i) = y$; (ii) for all $i \in \{X_2 \cup L \cup Y_2 \cup J\}$, $t(R'_i) \in \bar{U}(R_k, x)$ if $X_2 \neq \emptyset$ and $x < t(R'_i) < y$ otherwise, the relative order of alternatives in $M \setminus \{x, y\}$ is the same and strict, xR'_iy if and only if xR_iy; additionally, for any $i \in X_2$, $\bar{U}(R'_i, x) \subseteq \bar{U}(R_k, x)$;[13] (iii) for all $z \in M$, $w \in M^c$, zP'_iw; and (iv) the relative order of the alternatives in M^c is the same and is strict.

Case (b): $|X| > |Y_1|$. Let $J \subset Y_2$ with $|J| = |X| - |Y_1|$. If X_2 is not empty, let $k \in X_2$ be one of the agents for which $U(R_k, x) \subseteq U(R_i, x)$ for any $i \in X_2$. Let R' be a preference profile in $\mathcal{D}^{S,>}$ defined as follows: (i) for all $i \in X_1$, $t(R'_i) = x$, for all $i \in Y_1$, $t(R'_i) = y$ and for all $j \in J$, $t(R'_j) = y$; (ii) for all $i \in \{X_2 \cup L \cup Y_2 \setminus J\}$, $t(R'_i) \in \bar{U}(R_k, x)$ if $X_2 \neq \emptyset$ and $x < t(R'_i) < y$ otherwise, the relative order of alternatives in $M \setminus \{x, y\}$ is the same and strict, xR'_iy if and only if xR_iy; additionally, for any $i \in X_2$, $\bar{U}(R'_i, x) \subseteq \bar{U}(R_k, x)$;[14] (iii) for all $z \in M$, $w \in M^c$, zP'_iw; and (iv) the relative order of the alternatives in M^c is the same and is strict.

Before proceeding with the two cases, we state three facts that apply to them.

Fact 1: For any $B \in \mathcal{B}$ such that there exists $z \in B$ with $z > x$, $CW(R, B) > x$. This follows by assumption that $|X| < |Y|$.

Fact 2: For any $B \in \mathcal{B}$ such that $B \subseteq M^c$, $CW(R', B) \neq \emptyset$. (This follows since in R', the relative order of the alternatives in M^c is the same and strict.)

Fact 3: For any $B \in \mathcal{B}$ such that $B \subseteq M^c \cup \{\ell\}$ and $\ell \in M \cap B$, $CW(R', B) = \{\ell\}$. (This is because for all $i \in N$, for all $\ell \in M$, $w \in M^c$, $\ell P'_iw$.)

Now we show that $R' \in \mathcal{D}^{S,>}_{C\mathcal{B}}$ for any $\mathcal{B}$ for Case (a) and for Case (b).

[13] Note that from (ii), for any $i \in Y_2 \cup J$, for all $v \in M \setminus \{x, y\}$, vP'_iy.

[14] Note that from (ii), for any $i \in Y_2 \setminus J$, for all $v \in M \setminus \{x, y\}$, vP'_iy.

For any $B \in \mathcal{B}$ that satisfies conditions in Fact 2 or 3, there exists a strong Condorcet winner at (R', B) in both cases as shown in the facts. Consider any agenda $B \in \mathcal{B}$ not described in Facts 2 and 3. Then, there must exist $z, w \in M \cap B$ and $CW(R', B) \neq \emptyset$ implies $CW(R', B) \in M \cap B$ by condition (iii) in the definition of R' in both cases.

Case (a): $|X| \leq |Y_1|$. Note that by construction of R', $|X| = |\{i \in N : xP_i'y\}| = |\{i \in N : t(R_i') = y\}| = |Y_1| - |J|$ and $Y_2 \cup J$ is not empty.

Consider any $B \in \mathcal{B}$. Note that if $B \cap M = \{x, y\}$, $CW(R', B) = y$ trivially. If $B \cap M \neq \{x, y\}$, consider the unanimously most preferred alternative $w \in B \cap (x, y)$ according to R' for any $i \in X_2 \cup L \cup Y_2 \cup J$. We first show that w defeats any alternative $s > w$ by majority at R'. By definition of R', $|X_1 \cup X_2 \cup L \cup Y_2 \cup J| = |\{i \in N : wP_i's\}| > |\{i \in N : sP_i'w\}| = |Y_1 \setminus J|$. Also, w defeats any alternative $s \in B$ such that $s < w$ by majority at R', since by definition of R', $|Y_1 \setminus J \cup X_2 \cup L \cup J \cup Y_2| = |\{i \in N : wP_i's\}| > |\{i \in N : sP_i'w\}| = |X_1|$. Therefore, $CW(R', B) = w$.

Case (b): $|X| > |Y_1|$.

Note that by construction of R', $|X| = |\{i \in N : xP_i'y\}| = |\{i \in N : t(R_i') = y\}| = |Y_1| + |J|$ and since $|Y| > |X|$ and $|X| > |Y_1|$, we have that $Y_2 \neq \emptyset$.

Consider any $B \in \mathcal{B}$. Note that if $B \cap M = \{x, y\}$, $CW(R', B) = y$ trivially. If $B \cap M \neq \{x, y\}$, consider the unanimously most preferred alternative $w \in B \cap (x, y)$ according to R' for any $i \in X_2 \cup L \cup Y_2 \setminus J$. We first show that w defeats any alternative $s > w$ by majority at R'. By definition of R', $|X_1 \cup X_2 \cup L \cup Y_2 \setminus J| = |\{i \in N : wP_i's\}| > |\{i \in N : sP_i'w\}| = |Y_1 \cup J|$. Also, w defeats any alternative $s \in B$ such that $s < w$ by majority at R', since by definition of R', $|Y_1 \cup X_2 \cup L \cup J \cup Y_2 \setminus J| = |\{i \in N : wP_i's\}| > |\{i \in N : sP_i'w\}| = |X_1|$. Therefore, $CW(R', B) = w$.

Now, we show that R' satisfies parts 1 and 2 of Definition 6.

Part 1 For any $i \in N$, $L(R_i, x) \subseteq L(R_i', x)$ and $\overline{L}(R_i, x) \subseteq \overline{L}(R_i', x)$ and $xR_i'y \iff xR_iy$.

Case (a): $|X| \leq |Y_1|$. Note that for all $i \in N$, $xR_i'y$ if and only if xR_iy. Also, for all $i \in N$ and for all $w \in M^c$, $xP_i'w$. Moreover, for any $i \in Y \cup L$ and for all $z \in M \setminus \{x, y\}$, zP_ix. Therefore, for none of these agents, x gets worse relative to any alternative in $M \setminus \{x, y\}$ at R'. Additionally, for any $i \in X_2$, $\bar{U}(R_i', x) \subseteq \bar{U}(R_k, x)$ and for all $i \in X_1$, $t(R_i') = x$. Thus, part 1 is satisfied by construction of R'.

Case (b): $|X| > |Y_1|$. Note that for all $i \in N$, $xR_i'y$ if and only if xR_iy. Also, for all $i \in N$ and for all $w \in M^c$, $xP_i'w$. Moreover, for any $i \in Y \cup L$ and for all $z \in M \setminus \{x, y\}$, zP_ix. Therefore, for none of these agents, x gets worse relative to any alternative in $M \setminus \{x, y\}$ at R'. Additionally, for any $i \in X_2$, $\bar{U}(R_i', x) \subseteq \bar{U}(R_k, x)$ and for all $i \in X_1$, $t(R_i') = x$. Thus, part 1 is satisfied by construction of R'.

Part 2 There exists a permutation μ of A such that $x = \mu(y)$, $y = \mu(x)$, $\mu(R') \in \mathcal{D}_{\mathcal{CB}}^{S,>}$.

Denote $d(\ell, k)$ as the number of alternatives that are contained in the integer interval (ℓ, k) plus one and let $m = \{m_1, m_2\}$ be the median set in $M \subseteq A$.

We now define μ of A as follows: If m is a singleton, then $\mu : A \to A$ is such that (i) $z = \mu(z)$, for all $z \in M^c \cup \{m\}$ (ii) and $k = \mu(\ell)$ if and only if $d(m, k) = d(m, \ell)$ for all pairs $\ell, k \in M \setminus \{m\}$. If $m_1 \neq m_2$ are the median points, then (i) $z = \mu(z)$, for all $z \in M^c$ (ii) $m_1 = \mu(m_2)$ and $m_2 = \mu(m_1)$ and for all pairs $\ell, k \in M$ such that $k < m_1$ and $\ell > m_2$ and $d(m_1, k) = d(m_2, \ell)$, then $k = \mu(\ell)$ and $\ell = \mu(d)$.

It is immediate to check that, for any $B \in \mathcal{B}$, we have that $CW(\mu(R'', B))$. Therefore, $\mu(R') \in \mathcal{D}_{\mathcal{CB}}^{S,>}$.

Moreover, there exists a permutation ρ of N such that $\widehat{R} = \mu(R')_\rho \in \mathcal{D}_{\mathcal{CB}}^{S,>}$ and for any $i \in N$, $L(\widehat{R}_i, y) \subseteq L(R_i', y)$, $\overline{L}(\widehat{R}_i, y) \subseteq \overline{L}(R_i', y)$.

It is immediate to check that, for any $B \in \mathcal{B}$, we have that $CW(\widehat{R}, B) = CW(\mu(R'), B)$. Therefore, $\widehat{R} \in \mathcal{D}_{\mathcal{CB}}^{S,>}$.

Case (a): $|X| \leq |Y_1|$.

By construction of R', y is the unique worst alternative in M for any $j \in Y_1$ at $\mu(R')$ and for any $i \in X$ at R_i'. Since $|X| \leq |Y_1|$, for $|X|$ agents $j \in Y_1$, there exists $i \in X$ such that $L(\mu(R_j'), y) = L(R_i', y) = M^c \cup \{y\}$ and $\overline{L}(\mu(R_j'), y) = \overline{L}(R_i', y) = M^c$. Symmetrically, by construction of R', y is the unique best alternative in A at R' for any $j \in Y_1 \cup J$ and at $\mu(R'_i)$ for any $i \in X$. Since $|X| \leq |Y_1 \cup J|$, for any $i \in X$, there exists $j \in Y_1$ such that $L(\mu(R'_i), y) = L(R'_j, y) = A$, $\overline{L}(\mu(R'_i), y) = \overline{L}(R'_j, y) = A \setminus \{y\}$.

By construction of R', for any $\ell \in L$, $L(\mu(R'_\ell), y) = L(R'_\ell, y) = M^c \cup \{x, y\}$, $\overline{L}(\mu(R'_\ell), y) = \overline{L}(R'_\ell, y) = M^c$.

By construction of R', for any $j \in Y_2$ and $|Y_1| - |X|$ agents left in Y_1, y is the unique worst alternative in M at $\mu(R')$ and $M^c \cup \{y\} = L(\mu(R'_j), y) \subset L(R'_j, y)$, $M^c = \overline{L}(\mu(R'_j), y) \subset M^c \cup \{x\} \subseteq \overline{L}(R'_j, y)$.

We now define ρ of N as follows: (i) $j = \rho(i)$, for $|X|$ agents $j \in Y_1$ and any $i \in X$; (ii) $i = \rho(j)$, for $|X|$ agents $j \in Y_1$ and any $i \in X$; and (iii) $k = \rho(k)$, for any $k \in L \cup Y_2$ and the remaining $|Y_1| - |X|$ agents in Y_1.

Case (b): $|X| > |Y_1|$. By construction of R', y is the unique worst alternative in M at $\mu(R')$ for any $j \in Y_1 \cup J$ and at R'_i for any $i \in X$. Since $|X| = |Y_1 \cup J|$, for any $j \in Y_1 \cup J$, there exists $i \in X$ such that $L(\mu(R'_j), y) = L(R'_i, y) = M^c \cup \{y\}$, $\overline{L}(\mu(R'_j), y) = \overline{L}(R'_i, y) = M^c$. Symmetrically, by construction of R', y is the unique best alternative in A at R' for any $j \in Y_1 \cup J$ and at $\mu(R'_i)$ for any $i \in X$. Since $|X| = |Y_1 \cup J|$, for any $i \in X$, there exists $j \in Y_1 \cup J$ such that $L(\mu(R'_i), y) = L(R'_j, y) = A$, $\overline{L}(\mu(R'_i), y) = \overline{L}(R'_j, y) = A \setminus \{y\}$. By construction of R', for any $\ell \in L$, $L(\mu(R'_\ell), y) = L(R'_\ell, y) = M^c \cup \{x, y\}$, $\overline{L}(\mu(R'_\ell), y) = \overline{L}(R'_\ell, y) = M^c$. By construction of R', for any $j \in Y_2 \setminus J$, y is the unique worst alternative in M at $\mu(R')$ and $M^c \cup \{y\} = L(\mu(R'_j), y) \subset L(R'_j, y)$, $M^c = \overline{L}(\mu(R'_j), y) \subset \overline{L}(R'_j, y) = M^c \cup \{x\}$.

We now define ρ of N as follows: (i) $j = \rho(i)$, for any $j \in Y_1 \cup J$ and any $i \in X$; (ii) $i = \rho(j)$, for any $j \in Y_1 \cup J$ and any $i \in X$ and (iii) $k = \rho(k)$, for any $k \in L \cup Y_2 \setminus J$. ■

From now on, the Appendix is devoted to make a few remarks on the relation between Condorcet consistency and pairwise justifiability with two well-known properties of social choice functions: strategy-proofness and Maskin monotonicity, for the case of fixed agenda and variable profiles introduced in Subsection 5.2.[15]

Definition Let $\mathcal{D} \subseteq \mathcal{R}^n$ be a Cartesian preference domain. A collective choice function $C : \mathcal{D} \to A$ is *strategy-proof* on $\mathcal{D}$ if for any $i \in N$, any preference profile $R \in \mathcal{D}$, and any agent i's preference R'_i, we have that $C(R) R_i C((R'_i, R_{N \setminus \{i\}}))$.

Note that $R_{N \setminus \{i\}}$ refers to the preferences of all agents in N but i.

Definition Let $\mathcal{D} \subseteq \mathcal{R}^n$ be the preference domain. A collective choice function $C : \mathcal{D} \to A$ satisfies *Maskin monotonicity* on $\mathcal{D}$ if for any pair of preference profiles $R, R' \in \mathcal{D}$ such that for each agent $i \in N$, $\left[C(R) R_i z \Rightarrow C(R) R'_i z\right]$, then $C(R') = C(R)$.

First, we show that strategy-proofness and pairwise justifiability are independent when agent's preferences have indifferences. Example 8 shows a strategy-proof social choice function that violates pairwise justifiability. Example 9 shows a social choice function that satisfies pairwise justifiability but it violates strategy-proofness.

EXAMPLE 8. Let $N = \{1, 2\}$ and $A = \{x, y, z, w\}$. The set of admissible preference profiles is $\mathcal{D} = \{R_1, R'_1\} \times \{R_2, R'_2\}$ where R_1: $y I_1 w P_1 x I_1 z$ and R'_1: $y P'_1 w P'_1 x I'_1 z$, R_2: $z I_2 y P_2 x I_2 w$ and R'_2: $x I'_2 w P'_2 z I'_2 y$. Define C such that $C(R_1, R_2) = x$, $C(R_1, R'_2) = w$, $C(R'_1, R_2) = z$, and $C(R'_1, R'_2) = y$. It is easy to check that C is strategy-proof. However, C violates pairwise justifiability. To show the latter, take (R_1, R_2) and (R'_1, R'_2) and observe that no alternative improves with respect to $C(R_1, R_2) = x$ for no agent (equivalently, for any agent i, the lower contour set at x from R_i to R'_i weakly increases). Moreover, no agent is indifferent between x and y under (R_1, R_2): $y P_1 x$ and $y P_2 x$.

EXAMPLE 9. Let $N = \{1, 2\}$ and $A = \{x, y, z, w\}$. The set of admissible preference profiles is $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$, $\mathcal{D}_1 = \mathcal{D}_2 = \{R, R'\}$ where R: $x I z P y P w$ and R': $w I' x P' y I' z$. Define C such that $C(R_1, R_2) = y$, $C(R_1, R'_2) = C(R'_1, R_2) = x$, and $C(R'_1, R'_2) = w$. We can check that C satisfies pairwise justifiability. However, C violates strategy-proofness. To show the latter,

[15] See Barberà (2011) for a historical survey on strategy-proofness and Maskin (1999) for his well-known property.

take (R_1, R_2) and observe that agent 1 would strictly gain by saying R_1' instead of R_1 since $C(R_1', R_2) = xP_1y = C(R_1, R_2)$. To show that C satisfies pairwise justifiability, one must consider any pair of profiles with different outcome and check that either part 1 or 2 of the condition holds. In this example, 10 comparisons are required. For the sake of illustration, we prove it only for one pair of profiles: $R = (R_1, R_2)$ and $\widetilde{R} = (R_1', R_2)$. Observe that from R to $\widetilde{R}$, there is an agent, 1, and an alternative, w, such that $C(R) = yP_1w$ and $wP_1'y$.

Second, we prove that in subdomains of $\mathcal{D}_{CA}$, Condorcet consistency implies strategy-proofness (Proposition 4) but not Maskin monotonicity (Example 10). Finally, by means of Example 11, we show that in subdomains of $\mathcal{D}_{CA}$, strategy-proofness does not imply Condorcet consistency.

Proposition 4. *Let $\mathcal{D} \subseteq \mathcal{R}^n$ be a subset of preference profiles. Any Condorcet consistent collective choice function C on $\mathcal{D}$ satisfies strategy-proofness on $\mathcal{D}'$ where $\mathcal{D}' \subseteq \mathcal{D}_{CA}$ and has a Cartesian product structure.*

Proof. Let C be Condorcet consistent on $\mathcal{D}'$ where $\mathcal{D}' \subseteq \mathcal{D}_{CA}$ and $\mathcal{D}'$ has a Cartesian product structure. We prove that C is strategy-proof. By contradiction, suppose that there exist two preference profiles $R, R' \in \mathcal{D}_{CA}$, such that $R' = (R_i', R_{N\setminus\{i\}})$, $C(R) = x$, $C(R') = y$ and for some agent i, yP_ix. Since $C(R) = x$ is the strong Condorcet winner, then $C(R') \neq y$ and since if x defeats y by majority at R, it also defeats it at R'. □

Example 10. Consider $N = \{1, 2, 3\}$, $A = \{x, y\}$, and $\mathcal{D} = \{R, R'\}$. Let R be such xP_1y, xP_2y, and yP_3x and R' such that $xI_1'y$, $xI_2'y$, and $yP_3'x$. The strong Condorcet winner at R and R' are x and y, respectively. Any Condorcet consistent rule C must be such that $C(R) = x$ and $C(R') = y$. If C satisfied Maskin monotonicity, we should have that $C(R') = C(R)$ that is not the case.

Example 11. Consider $N = \{1, 2, 3\}$, $A = \{x, y, z\}$, and $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2 \times \mathcal{D}_3$ being for each $i \in N$, $\mathcal{D}_i = \{R_i, R_i', R_i''\}$ such that xP_iyP_iz, $xI_i'yI_i'z$, and $yP_i''xP_i''z$. Let the outcome of the collective choice function C be defined as the element in each cell in the following tables. Note that in the rows, we represent preferences of agent 1, in columns preferences of agent 2, and in each table represents a preference of agent 3.

R_3	R_2	R_2'	R_2''
R_1	x	x	x
R_1'	x	y	y
R_1''	x	y	y

,

R_3'	R_2	R_2'	R_2''
R_1	x	$\mathbf{y}$	y
R_1'	y	y	y
R_1''	y	y	y

,

R_3''	R_2	R_2'	R_2''
R_1	x	y	y
R_1'	y	y	y
R_1''	y	y	y

The reader can check that C is strategy-proof on $\mathcal{D}$, thus on $\mathcal{D}_{CA} = \mathcal{D}\setminus\{(R_1', R_2', R_3'), (R_1, R_2', R_3''), (R_1, R_2'', R_3'), (R_1', R_2, R_3''), (R_1', R_2'', R_3), (R_1'', R_2, R_3'), (R_1'', R_2', R_3)\}$. However, C violates Condorcet consistency since at preference profile (R_1, R_2', R_3'), $C((R_1, R_2', R_3')) = y$ but the Condorcet winner exists and is x.

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