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Mathematical Knowledge of Early Algebra Exhibited by Pre-Service Early Childhood Education Teachers

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Abstract
This study analyzes the mathematical knowledge to teach early algebra exhibited by pre-service early childhood education teachers, from the perspective of the Mathematical Knowledge for Teaching (MKT) model. The research adopts a mixed exploratory-descriptive methodological approach, based on the application of the MKT-early algebra questionnaire (3–6), consisting of six open-ended items that place teachers in various teaching situations reflecting the knowledge that characterizes early algebra at this stage of schooling. The analysis of the answers given by the pre-service teachers of early childhood education revealed a level general of insufficient mathematical knowledge, with the common content knowledge exhibiting fewer limitations compared to the other subdomains that comprise the model, and the horizon content knowledge the weakest. We conclude that it is necessary to offer teacher training programs that deepen the didactics of early algebra and provide tools to further the effective teaching of this content block in early childhood education.

Keywords Early algebra · Early childhood education · Mathematical knowledge for teaching · Pre-service teacher

Introduction

A growing research agenda in early algebra suggests that early childhood education children can reasoning algebraically (e.g., Lenz, 2022; Lüken & Sauzet, 2020; Rittle-Johnson et al., 2015). Kieran et al.'s (2016) studies have focused primarily on explaining students' opportunities to explore and discern mathematical

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relationships, patterns, and arithmetic structures through processes of observation, conjecture, generalization, representation, justification, and communication (e.g., Ayala-Altamirano & Molina, 2020; Pinto et al., 2022; Torres et al., 2024).

On a related topic, the "algebraization" of the curriculum, a term proposed by Kaput (2000) to refer to the integration of algebraic reasoning or thinking throughout schooling, has begun to be incorporated into some contemporary early childhood education curricula (e.g., Australian Curriculum, Assessment And Reporting Authority [ACARA], 2022; Ministry of Education, Republic of Singapore, 2013; National Council of Teachers of Mathematics [NCTM], 2000), since they have integrated the teaching of algebra as a content block starting with this stage of education (Pincheira & Alsina, 2021a).

The latter means "teachers can help students build a solid foundation of understanding and experience as a preparation for more sophisticated work in algebra in the middle grades and high school" (NCTM, 2000, p. 39). However, to ensure that children benefit from such learning experiences involving early algebra, early childhood education teachers need to have specific knowledge (Mosvold et al., 2011), since they are a key agent in eliciting the development of algebraic thinking as a capacity to make and express generalizations (Kaput, 2008; Pinto et al., 2022).

Despite the interest in researching the development of algebraic thinking in early childhood education, and advances in early algebra in curricular terms, research on how to guide teachers towards the effective teaching of early algebra at this school stage (3–6 years old) is not as advanced. Accordingly, the literature notes the need to conduct studies that analyze the mathematical knowledge of early algebra of early childhood education teachers (e.g., Cabral et al., 2021; Pincheira & Alsina, 2021b), since there is little evidence on the development of this knowledge to undertake the teaching of early algebra (Walkoe et al., 2022). Consequently, studies are needed that can be used to evaluate and track key aspects of mathematical knowledge in order to carry out the instructional process involved in this content block.

Lane et al. (2015) note that the knowledge of teachers is directly related to student learning. Based on this, we posit the question: what mathematical knowledge does an early childhood education teacher mobilize in order to teach early algebra?

Given this interrogatory, the objective of our study is to analyze the mathematical knowledge of pre-service teachers for teaching early algebra in early childhood education. To achieve this goal, we take the perspective of Ball et al. (2008) and use the Mathematical Knowledge for Teaching (MKT) model as an analytical tool. The data obtained will provide the starting point to determine those central aspects that should be considered in the initial training of early childhood education teachers to teach early algebra.

Theoretical Foundation

Mathematical Knowledge for Teaching

In the field of teacher knowledge, the research contributions proposed by Shulman (1986, 1987) showed a lack of emphasis on both teacher training and evaluation, as well as on effective teaching practices. Shulman (1986) introduced the concept of content knowledge, referring to “the amount and organization of knowledge per se in the mind of the teacher” (p. 9), and pedagogical knowledge, defined as a “particular form of content knowledge that embodies the aspects of content most germane to its teachability” (p. 9). By distinguishing between the two concepts, this author intended to bridge the gap between content and pedagogy.

Ball et al. (2008), in an effort to refine and empirically validate the notions proposed by Shulman (1986, 1987), developed a model of knowledge specific to mathematics teachers, called Mathematical Knowledge for Teaching (MKT). This model has been developed as an analytical tool of teacher knowledge and is defined as “the mathematical knowledge that teachers uses in the classroom to produce instruction and student growth” (Hill et al., 2008, p. 374); that is, it refers to the multiple dimensions of knowledge on which a teacher relies when implementing the teaching of mathematics.

The MKT model represents a map of the mastery of mathematical comprehension and ability that considers two main constructs: knowledge of the subject and pedagogical knowledge of the content. On the one hand, knowledge of the subject integrates: common knowledge of content (CCK), defined as “knowledge that is used in the work of teaching in ways in common with how it is used in many other professions or occupations that also use mathematics” (Hill et al., 2008, p. 377); knowledge of specialized content (SCK), which is “mathematical knowledge that allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems” (Hill et al., 2008, pp. 377–378); and horizon content knowledge, defined as “an awareness of how mathematical topics are related over the span of mathematics included in the curriculum” (Ball et al., 2008, p. 403).

And on the other hand, pedagogical knowledge of the content considers: knowledge of content and students (KCS), defined as “knowledge that combines knowing about students and knowing about mathematics” (Ball et al., 2008, p. 401), that is, the knowledge that can be used to anticipate and interpret how students think when confronted with a certain mathematical task; knowledge of content and teaching (KCT), described as that which “combines knowing about teaching and knowing about mathematics” (Ball et al., 2008, p. 401) and which refers to the mathematical knowledge of how to design the instruction, that is, how to use different methods and procedures in order to develop the instruction; and finally, knowledge of the content and curriculum related to content knowledge as it pertains to the curriculum designed for each educational level in the area of mathematics.

Although the MKT model, described earlier, was at one time a significant advancement in characterizing the knowledge that a teacher should have to teach mathematics, Ball et al. (2008) note that “It is not always easy to discern where one of our categories divides from the next, and this affects the precision (or lack thereof) of our definitions” (p. 403). Thus, one of the main limitations of the model is to discern common knowledge from specialized knowledge in specific cases.

Mathematical Knowledge of Early Algebra in Early Childhood Education Teachers

Mathematical Knowledge for Teaching [MKT] (Ball et al., 2008) is an analytical model that has gained international relevance and whose purpose is to analyze the knowledge of mathematics teachers by considering various components. In relation to knowledge for teaching algebra, the systematic review conducted by Pincheira and Alsina (2021b) shows the links between the two aspects, based on 17 papers published in Web of Science and Scopus during the period 2010–2021 that analyze the knowledge for teaching algebra from an MKT perspective. On the one hand, the data from the studies show advances in the conceptualization of mathematical knowledge for teaching algebra during the period analyzed, especially in Primary Education teachers and in the specialized knowledge of content; on the other hand, there is a predominance of studies that analyze functional thinking.

Some recent studies have begun to analyze the mathematical knowledge that early childhood education teachers have of general aspects of early algebra typical of this stage of schooling, focusing mainly on knowledge of mathematical patterns (Pincheira & Alsina, 2021b).

For example, Bair and Rich (2011), in a longitudinal study with more than 5000 pre-service early childhood and primary education teachers, analyzed the knowledge of specialized content for teaching in relation to algebraic reasoning and number sense. These authors identified problems exemplifying the nature of the relationships between quantities and establishing connections between the representations of a number sequence. Similarly, Noviyanti and Suryadi (2019) evaluated the basic mathematical knowledge of 35 in-service early childhood education teachers on patterns and mathematical relationships. The results reveal limitations in content knowledge in both constructs. Cabral et al. (2021), from a broad perspective on teacher training in algebra, carried out a training experience with two pairs of Early Childhood Education teachers in training, where they analyzed the mathematical knowledge about repetitive patterns and the ability to perceive algebraic thinking of children in Early Childhood Education. The results reveal difficulties in understanding repetitive patterns as a mathematical object, however, they address relevant aspects of children's algebraic thinking, presenting some limitations in their interpretation.

Pincheira and Alsina (2022b) analyzed the specialized content knowledge and the knowledge of content and teaching of 18 pre-service early childhood education teachers when designing mathematical tasks on repetition patterns. The results reveal problems in the specialized knowledge to identify theoretical concepts of a task on patterns, demonstrate rules of sequence formation and poor management of teaching situations in which to apply the study of patterns. They also indicate deficiencies in the knowledge of content and teaching to propose teaching strategies that promote an understanding of patterns and the use of early algebraic language. Unlike the previous study, this research pursues a broader objective, since it considers all the subdomains of knowledge of the MKT model, delving not only into repetition patterns, but also into other content areas that characterize early algebra in childhood education, such as different types of relationships and change (Pincheira & Alsina, 2021a).

In summary, the data on the mathematical knowledge mobilized by pre- and in-service early childhood education teachers are scarce, and reflect a limited knowledge to promote the effective teaching of early algebra at this stage of schooling.

Methodology

In keeping with the purpose of the research that, as indicated, consists of analyzing the mathematical knowledge of pre-service teachers for teaching early algebra in early childhood education, we employed a mixed methodological approach of an exploratory-descriptive type (Creswell, 2014) because, on the one hand, it addresses a subject that has received limited attention in the literature and, on the other, it seeks to specify the MKT characteristics of early childhood teachers in the different subdomains that comprise it.

Participants and Context

The study involved 60 students majoring in early childhood education in a university in Spain. The sample was selected by considering a non-probabilistic sampling of an accidental or causal nature (Creswell, 2014), since the selection criterion was determined by the possibility of joining this group.

The ages of the participants ranged from 18 to 25, with 59 women (98.3%) and 1 man (1.7%). As for their previous education, 49 (81.6%) of them completed high school, 25 (41.7%) completed vocational training, and 14 (23.3%) completed both.

At the time of the study, in 2022, the participants were in the second year of their studies, out of a total of four, and were taking the "Learning mathematics" course. In general, in this first course in the field of mathematics education, pre-service teachers receive general training on early childhood mathematical education (skill-based planning and managing of teaching practices; presence of mathematics in the kindergarten curriculum). They then study the content, assessment indicators, resources and strategies for teaching early algebra to students aged 3 to 6 years, and other content blocks, such as numbering and calculation, geometry, measurement, and statistics and probability.

At the beginning of the research, the pre-service teachers were in the middle of the "Learning Mathematics" course and had received training in early algebra for the first time, considering both the main contemporary contributions of research into early childhood mathematics education and the most advanced curricula on this subject. This is because, for Spain, there is a critical analysis of the early childhood education curriculum (MEFP, 2022), given the mismatches between the proposed contents in comparison with both the research contributions (Alsina, 2022b) and the international curricular proposals (e.g., ACARA, 2022; NCTM, 2000). In any case, at the time of the research, they did not yet have any practical experiences in the early childhood education classroom.

Design and Procedure

The data were obtained by administering the MKT-early algebra (3–6) questionnaire described in Pincheira and Alsina (2022a), where the validity and reliability of the instrument were analyzed, yielding a Cronbach alpha of 0.72. This instrument consists of six open-ended exercises (Table 1) that place pre-service teachers in various teaching situations that can provide an integrated analysis of the mathematical contents that characterize early algebra in early childhood education (Pincheira &

Table 1 Items that make up the MKT early algebra questionnaire (3–6)

Item 1:

A teacher proposes to 4-year-olds playing the game "We are detectives". Below is an excerpt from the planning activity:

The children are separated into groups and observe how some materials in the room are arranged. They answer questions such as: What are these materials used for? What do they have in common? How are they different from the materials of the other piece of furniture? Why do they think they are grouped together? How are they alike? Then, each group shares their ideas and, together, they discover what criteria were used to arrange them. Finally, they establish agreements to propose new criteria to arrange the materials and do their best to implement them to reorganize the classroom.

Questions:
a. What mathematical content should the children employ to take part in the task proposed by the teacher? Fuente
b. Considering the kindergarten curriculum, what could be the goal of the task?
c. With what more advanced concepts of the school curriculum is the content in the task related?

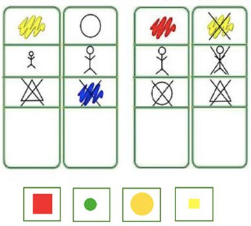
Item 2:

A teacher proposes the following task to 4-year-olds:
"Use a line to join the elements of the row that belong to each group"

Questions:
a. What mathematical content should the children use to provide a correct solution to the problem posed?
b. Describe the possible difficulties children would face to correctly solve the task.
c. What teaching strategies would you use to help children who had problems solving the task?

Item 3:

A teacher shows 5-year-olds the following attribute cards and instructs them to read the attributes of each card and associate them to the corresponding piece at the bottom:



Questions:
a. What mathematical content should children use to provide a correct solution to the task?
b. Describe the possible difficulties children would face to correctly solve the task.
c. What other resource would you use for the children to perform this type of task? Explain how you would use it and justify your choice.
d. Considering the kindergarten curriculum, what could be the goal of the task?

Item 4:

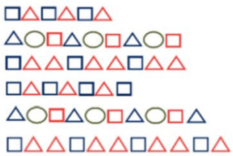
A teacher shows 3-year-olds a set of *Multilink* cubes. The goal of the activity is to "Build a simple series through the free manipulation of the material provided".
The following describes a scenario that occurs involving a girl:

Girl: A tower! A green one comes next
Teacher: Why is a green one next?
Girl: Because it goes green, orange, green, orange
Teacher: What happened in the middle of the tower? (The teacher points out the mistake)

Questions:
a. Following the series described verbally by the girl, what cube should be placed in the 21st place? Explain how you got your answer.
b. What mathematical content did the girl use to construct the serialization?
c. Describe the possible problems that led the girl to respond incorrectly.
d. What teaching strategies would you use to help the girl realize and correct her mistake? Explain your answer.

Item 5:

A teacher does the following task with the 5-year-olds: first, she prepares boxes with cutouts of triangles, squares and circles; she then presents the first series (P1), chooses a child and places two boxes in front of him/her, one with blue squares and the other with red triangles; then asks: "what comes next?"
With the series that follow, she continues this process with the children in the class.

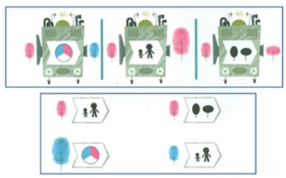


Questions:
a. Determine the unit of repetition (the pattern) of each series. Explain your answer.
b. Describe the possible difficulties children would face to correctly solve the task.
c. What teaching strategies would you use to help those children who had problems solving the task?
d. Considering the kindergarten curriculum, what could be the goal of the task?

Item 6:

An Early Childhood Education textbook proposes the following task for 5-year-olds:
"Draw the figure that comes out of the machine for changing qualities"

To do the task, the children have to look at the examples in the top box, which describe the change made by each operator - color, size, and shape respectively - and then draw a possible solution in the box at the bottom.



Questions:
a. What mathematical content should the children use to answer correctly?
b. Describe the possible difficulties children would face to correctly solve the task.
c. What teaching strategies would you use to help those children who had problems solving the task?
d. Considering the pre-school curriculum, what could be the goal of the task?

Alsina, 2021a). This characterization emerges from the analysis of various curricular approaches at the international level (e.g., ACARA, 2022; Ministry of Education, Republic of Singapore, 2013; NCTM, 2000, among others), and establishes some initial categories of knowledge: (a) establish relationships based on the recognition of attributes when experimenting with elements or objects, such as classifications, arrangements and correspondences, etc., (items 1, 2 and 3); (b) seriation based on repetition patterns, considering the identification, construction and representation of the pattern (items 4 and 5); and (c) description of qualitative changes related to changes in physical attributes, and quantitative changes related to changes in quantities (items 6). It is important to continue delving into these categories of knowledge, since they reflect the area of algebra content declared in contemporary early childhood education curricula (3 to 6 years old).

Table 2 shows the guidelines that provided a reference for designing each of the six items that comprise the questionnaire.

The items that pertain to CCK are related to the seriations with repetition patterns, and they ask to identify the specific position of a seriation (item 4a), as well as to determine the repetition unit of various seriations (item 5a). The items that delve into SCK propose situations for establishing relationships (items 1a, 2a and 3a), pattern seriations (item 4b) and description of qualitative changes (item 6a). These items call on exposing the mathematical contents that allow solving each task. Meanwhile, HCK delves into the most advanced concepts of the curriculum that involve the relationships that are established from the recognition of attributes (item 1c).

As concerns KCS, it investigates the possible difficulties that children may have in establishing object classification relationships (item 2b) and element comparisons (item 3b) through the recognition of attributes, as well as the difficulties they exhibit when creating patterns (item 4c), expanding seriations that do not end in a complete repetition unit (item 5b), and applying qualitative changes from a logical operator (item 6b). KCT is covered by requesting didactic strategies or resources to deal with different types of relationships (items 2c and 3c), working with repetition patterns (items 4d and 5c), and the notion of change (item 6c). Finally, KCC requests to expose the purpose of each task based on the curricular guidelines involving the representation of attributes of elements and collections, establishing classification,

Table 2 Guidelines for the MKT model used in the design of the items

Subdomain	Guideline
CCK	Solves algebraic tasks
SCK	Identifies what mathematical content children need to put into practice to solve the task.
HCK	Identifies the more advanced mathematical content of the school curriculum that is involved in the task
KCS	Determines the main difficulties children might have in learning early algebraic content
KCT	Indicates instructional strategies or resources they would use to teach the algebraic content
KCC	Determines what the goal of the task is, based on the preschool education school curriculum

grouping and comparison relationships (item 1b, 3d), seriations (item 5d) and change (item 6d).

The items on the questionnaire provide an insight into the mathematical knowledge involved in teaching early algebra in early childhood education, giving rise to a total of twenty-two questions that are based on the MKT model (Ball et al., 2008), allowing us to investigate the domains and subdomains that compose it, as shown in Table 3. These questions were taken and adapted from previous research that delved into the different subdomains of the model.

To assess the degree of adequacy of each item based on the subdomains of knowledge of the MKT model, a relevance analysis (Pincheira & Alsina, 2022a) was carried out based on the validation of the instrument through the judgment of experts with extensive experience in the field of Mathematics Education, and more specifically, with the MKT Model and the study of early algebra. Based on an evaluation guideline, 12 experts assigned ratings to each item (3: relevant; 2: unsure; 1: not relevant). The results show high scores for the items (Table 4). Therefore, the analysis carried out a priori confirms that the items designed in the questionnaire are relevant to measure different subdomains of the MKT model in relation to the teaching of early algebra in early childhood education (3–6 years old).

It is important to note that the questionnaire was administered in the context of a regular class in the training process of the participants (90-minute session), with the authorization and collaboration of the professor in charge of the "Learning mathematics" course. The researchers also obtained the informed consent of the participants, who collaborated and responded to the questionnaire voluntarily.

Data Analysis

After the data were gathered, the answers to the questionnaire provided by the pre-service early childhood education teachers were analyzed. The results provided an insight into the mathematical knowledge of early algebra of the teachers in question, from the perspective of the MKT model (Ball et al., 2008).

The analysis of the data considered quantitative and qualitative aspects. The former estimated the variable "degree of correctness of the answers" by assigning a score of 2 if the answer was correct, 1 if it was partially correct and 0 if it was incorrect. To assign these scores, criteria were established using an evaluation rubric based on the relevance of the answers, such that the maximum score on the questionnaire was 44 points and the minimum was 0 points. This rubric was submitted to the judgment of experts in didactics of mathematics and early algebra.

As for the qualitative analysis, the various answers given by the pre-service teachers were categorized using the content analysis technique (Krippendorff, 2013). Thus, methods of constant comparison (Corbin & Strauss, 2008) between the answers, categorizing the data as per the coding established by the rubric, allowed us to identify categories such as "types of errors in the answers", "justifications provided by the teachers", "problems finding the right answer", and others.

To ensure the reliability of the coding process, the answers to the questionnaire were successively reviewed in a cyclical and deductive manner, considering the

Table 3 Domains and subdomains of the MKT model to be evaluated in the items that make up the questionnaire

Items		Content knowledge			Pedagogical knowledge of the content		
		CCK	SCK	HCK	KCS	KCT	KCC
1	a. What mathematical content should the children employ to take part in the task proposed by the teacher?		x				
	b. Considering the kindergarten curriculum, what could be the goal of the task?						x
	c. With what more advanced concepts of the school curriculum is the content in the task related?			x			
2	a. What mathematical content should the children use to provide a correct solution to the problem posed?		x				
	b. Describe the possible difficulties children would face to correctly solve the task.				x		
	c. What teaching strategies would you use to help children who had problems solving the task?					x	
3	a. What mathematical content should children use to provide a correct solution to the task?		x				
	b. Describe the possible difficulties children would face to correctly solve the task.				x		
	c. What other resource would you use for the children to perform this type of task? Explain how you would use it and justify your choice.					x	
	d. Considering the kindergarten curriculum, what could be the goal of the task?						x
4	a. Following the series described verbally by the girl, what cube should be placed in the 21st place? Explain how you got your answer.	x					
	b. What mathematical content did the girl use to construct the serialization?		x				
	c. Describe the possible problems that led the girl to respond incorrectly.				x		
	d. What teaching strategies would you use to help the girl realize and correct her mistake? Explain your answer.					x	
5	a. Determine the unit of repetition (the pattern) of each series. Explain your answer.	x					
	b. Describe the possible difficulties children would face to correctly solve the task.				x		
	c. What teaching strategies would you use to help those children who had problems solving the task?					x	
	d. Considering the kindergarten curriculum, what could be the goal of the task?						x

Table 3 (continued)

Items		Content knowledge			Pedagogical knowledge of the content		
		CCK	SCK	HCK	KCS	KCT	KCC
6	a. What mathematical content should the children use to answer correctly?		x				
	b. Describe the possible difficulties children would face to correctly solve the task.				x		
	c. What teaching strategies would you use to help those children who had problems solving the task?					x	
	d. Considering the early childhood education curriculum, what could be the goal of the task?						x

CCK Common Content Knowledge, *SCK* Specialized Content Knowledge, *HCK* Horizon Content Knowledge, *KCS* Knowledge of Content and Students, *KCT* Knowledge of Content and Teaching, *KCC* Knowledge of Content and Curriculum

Table 4 Relevance of questionnaire items as validated by the judgment of experts

Item		Relevant (3)	Unsure (2)	Irrelevant (1)	Avg.	Item		Relevant (3)	Unsure (2)	Irrelevant (1)	Avg.
1	a)	11	0	1	2.8	4	b)	12	0	0	3
	b)	11	1	0	2.9		c)	12	0	0	3
	c)	11	1	0	2.9		d)	11	1	0	2.9
2	a)	11	1	0	2.9	5	a)	11	1	0	2.9
	b)	12	0	0	3		b)	12	0	0	3
	c)	11	1	0	2.9		c)	11	1	0	2.9
3	a)	12	0	0	3		d)	12	0	0	3
	b)	12	0	0	3	6	a)	12	0	0	3
	c)	11	1	0	2.9		b)	12	0	0	3
	d)	12	0	0	3		c)	11	1	0	2.9
4	a)	12	0	0	3		d)	12	0	0	3

evaluation rubric. Then, a triangulation was carried out based on continuous reviews of the responses, ending with a discussion, by the authors, of the disagreements involving the coding process until a consensus was reached.

Results

The following describes the data obtained, showing, first, an analysis of the total scores of the questionnaire and, second, an analysis of the results obtained in relation to the domains and subdomain of the mathematical knowledge for teaching.

Total Questionnaire Score

The total questionnaire scores were analyzed based on the degree of correctness of the answers given by the 60 pre-service early childhood education teachers.

The total scores on the questionnaire ranged from 8 to 34 points, with no maximum scores reported (44 points). The average score was 18 points, which is less than half of the maximum score, with a standard deviation of 5.0 points. Likewise, the percentage of right answers on the questionnaire was 41%.

Figure 1 shows that the median questionnaire score was low, 17.6, meaning the median was slightly closer to the first quartile. Therefore, the values of the total questionnaire scores were slightly more concentrated in the lower half of the box.

We also see that the amplitudes of the top and bottom "whiskers" are relatively similar. This indicates some similarity between the extremes of the distribution of total scores. We do, however, see the presence of an outlier in the top area of the box, corresponding to an extreme observation that deviates from the bulk of the data; in this case, it corresponds to a pre-service teacher who obtained a total score of 34 points out of a possible 44 points.

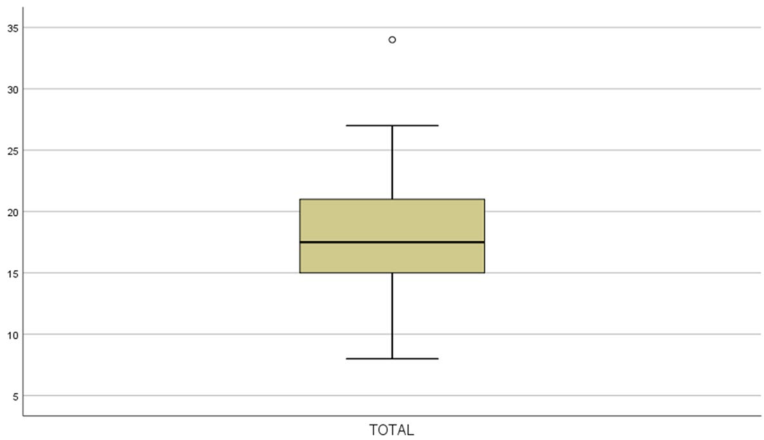


Fig. 1 Distribution of total scores and median score on the questionnaire

Comparison of the Domains and Subdomains of Mathematical Knowledge to Teach Early Algebra

To establish a comparison between the subdomains of mathematical knowledge to teach early algebra, the total scores for the questionnaire were recoded based on the type of knowledge based on a normalized scale of 0 to 100, since the number of items differs for each subdomain.

Figure 2 shows the scores obtained by the pre-service early childhood education teachers in the different subdomains of knowledge. In general, we see that the normalized scores obtained in the common content knowledge subdomain are higher than in the other knowledge subdomains. This is in contrast to the subdomain of mathematical horizon content knowledge, which yielded the lowest score on the questionnaire.

By considering more specifically the domain of knowledge of the subject, it is possible to observe that more than 50% of the pre-service teachers exhibit high normalized scores, above 75%, in the common content knowledge subdomain. Regarding the specialized content knowledge, the distribution of the normalized scores coincides with the knowledge of content and student subdomain, part of the pedagogical knowledge domain. In both subdomains, the normalized scores are concentrated in the upper area of the box, between 40% and 50%. Similarly, in the knowledge of the content and teaching, the median is closer to the third quartile, with the normalized scores clustering between 50% and 60%, in the top half of the box.

Finally, regarding the knowledge of the curriculum, 50% of the pre-service teachers failed to exceed 38% of the normalized scores.

As Fig. 2 shows, there are differences between the scores obtained for the different subdomains of mathematical knowledge. To delve into these differences and

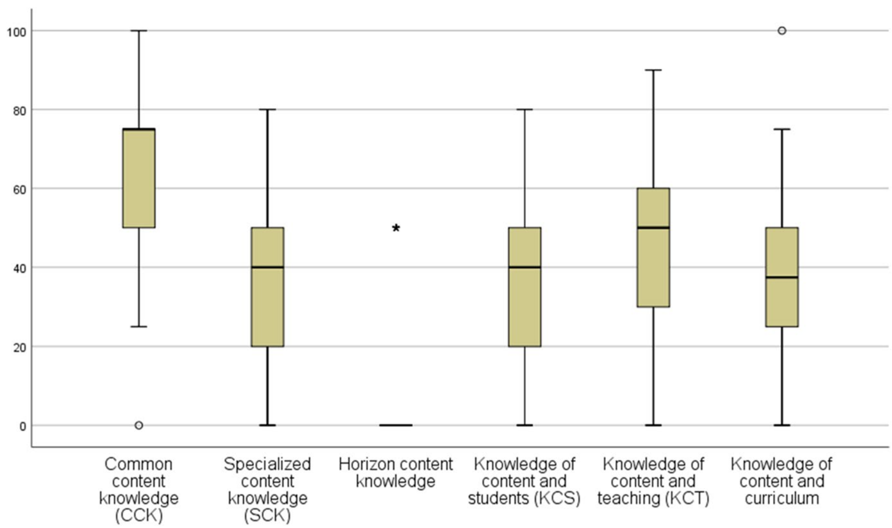


Fig. 2 Distribution of normalized scores by subdomains of mathematical knowledge

Table 5 Wilcoxon Signed-Rank test of the scores obtained in relation to the different subdomains of the MKT model

	CCK	SCK	HCK	KCS	KCT
SCK	0.000	-	-	-	-
HCK	0.000	0.000	-	-	-
KCS	0.000	0.000	0.000	-	-
KCT	0.000	0.000	0.000	0.041	-
KCC	0.000	0.000	0.000	0.897	0.358

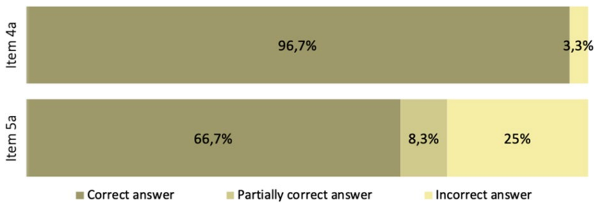


Fig. 3 Composition of the different types of answers for the common content knowledge by degree of correctness

determine if they are statistically significant, pairs of subdomains were compared by applying a non-parametric sign statistic for paired samples with a 95% confidence level, as seen in Table 5.

The Wilcoxon rank tests applied to the different subdomains of knowledge are mostly < 0.05. Therefore, the different subdomains of knowledge that were compared to one another show significant differences in their scores; whereas the comparison between knowledge of the curriculum and knowledge of content and students, as well as knowledge of the curriculum and knowledge of content and teaching, exhibit similar scores.

Analysis of the Subdomains of Mathematical Knowledge for Teaching

To delve into the subdomains of the mathematical knowledge of the participants, we analyzed the correctness of the answers to the MKT-early algebra questionnaire (3–6) and the explanations given in the answers to the different items that comprise it.

Common Content Knowledge

The set of items focused on evaluating the common content knowledge (4a and 5a) analyzes the answers the pre-service teachers gave when solving seriation tasks based on repetition patterns. In item 4a, the pre-service teachers were expected to identify the term in a series of two elements (green cube-orange cube) based on an indicated position, and in item 5a, to identify the repetition unit in six seriations proposed (P1 to P6) of two and three elements. The degree of correctness of the answers given for these items is shown in Fig. 3.

The percentage of correct answers in the common content knowledge block exceeds 66.7%. An analysis of the answers to item 4a shows a high percentage of correct answers (96.7%), meaning most of the pre-service teachers were able to specify and justify that a green cube belonged in position 21 of the series, for example, "in place 21 there would be a green cube, thinking that all the green cubes are odd and the orange cubes are even" (pre-service teacher 12); while only 3.3% answered incorrectly, stating it should be an orange cube.

In the case of item 5a, the partially correct answers (8.3%) are due to the lack of justification for the answer given. More specifically, the pre-service teachers noticed that in series P1 and P4, the repeating unit is blue square-red triangle, in series P2 and P5, it is blue triangle, gray circle and red square, and in series P3 and P6, it is blue square, red triangle, red triangle. However, they do not specify that the proposed series correspond to patterns of type AB, ABC and ABB, respectively. Meanwhile, 25% of the participants gave another answer or failed to accurately identify the repetition unit of the series.

Specialized Content Knowledge

To evaluate the specialized content knowledge, we focused on the mathematical content that pre-service teachers must be able to identify in order to solve certain situations involving algebraic teaching. This includes establishing relationships (classify, group, compare) based on the recognition of attributes, seriations with repetition patterns, and descriptions of qualitative changes. Accordingly, the questionnaire considers five items (1a, 2a, 3a, 4b and 6a) that pose questions such as, "What mathematical content should children use to correctly answer the task?" Fig. 4 shows the degree of correctness of the answers provided for these items.

The results show that items 1a and 2a, which correspond to the mathematical content related to establishing relationships by recognizing attributes, yielded a higher percentage of correct answers. However, this is not significant, since

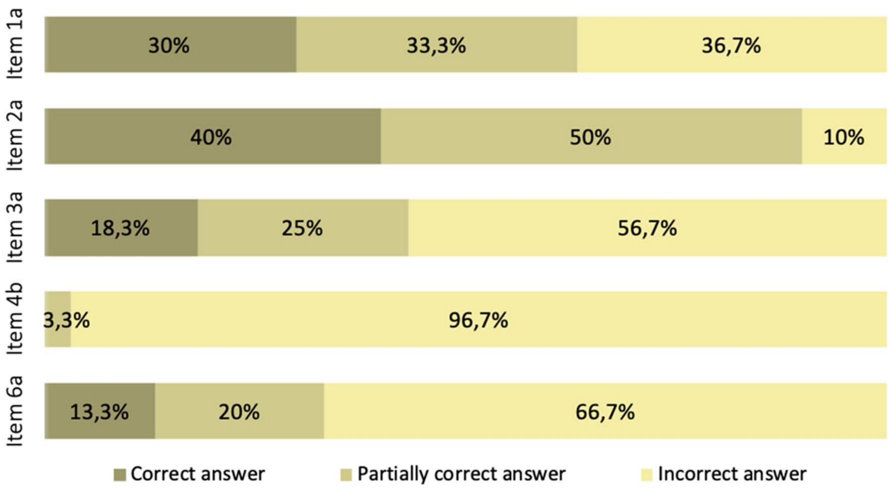


Fig. 4 Composition of the different types of answers for the specialized content knowledge by degree of correctness

the correct answers do not exceed 40%. Items 3a and 6a show greater limitations, since the number of correct answers is below 18.3%. In the case of item 4b, involving mathematical content on series with repetition patterns, it yielded a high percentage of incorrect answers (96.7%), being one of the most difficult items on the questionnaire.

In general, an analysis of the arguments presented shows that much of the content identified is generic. For example, many of the answers focus on content, such as algebra, early algebra, logic, mathematical logic, relationships and series. The pre-service teachers did not analyze the contents of the task they were provided. They also gave answers that did not correspond to mathematical content, but to mathematical processes, such as problem solving and representation. This thus reveals confusion on the part of the pre-service teachers between mathematical content and the mathematical processes involved in the teaching situations presented.

More specifically, unlike items 4b and 6a, the arguments for items 1a, 2a and 3a show a more detailed analysis of the mathematical content; for example, they mentioned aspects such as classification of elements, description of attributes and qualities, establishing similarities and differences based on a comparison of objects, grouping elements by two or three attributes, and more.

Finally, a low percentage (13.3%) managed to adequately identify in item 6a that the mathematical content associated with the task is related to direct logical operators and the representation of qualitative changes, while 20% only mentioned very generically that the content involves change, and a large percentage failed to identify the mathematical content of the task.

Mathematical Horizon Knowledge

Knowledge of the mathematical horizon was assessed using item 1c of the MKT-early algebra questionnaire (3–6). To analyze this knowledge, the participants were asked to establish links between the mathematical content involved in the teaching situation and others proposed in the extension of the curriculum, based on the question: to what other more advanced concepts in the curriculum can the content presented in the task be associated?

The teaching situation requires establishing relationships of classification and grouping of elements based on recognizing attributes (shape, color, size, etc.). Participants are thus expected to relate the content addressed in the task with the idea of class inclusion and the notion of part-whole, a key element in building the concept of number.

The results show that item 1c was one of the most difficult items in the questionnaire, since no correct answers were provided. 88.3% of participants responded incorrectly to the item, offering nonsensical arguments.

Only seven pre-service teachers (11.7%) gave a partially correct answer, for example, "we can relate it to elements that belong to a set or those that do not belong" (pre-service teacher 11), arguing that the content presented in the task is related to the sense of belonging or not.

Knowledge of Content and Students

To respond to the set of items focused on evaluating the knowledge of content and students, the pre-service teachers need to anticipate how the children think in relation to the errors or difficulties they might encounter when faced with a certain algebraic task.

To analyze this knowledge, the questionnaire proposes five items (2b, 3b, 4c, 5b and 6b) that investigate issues such as “describe the possible difficulties that the children would have correctly solving the task”. The degree of correctness of the answers given for these items is shown in Fig. 5.

In general, the results show that the percentage of correct answers is low, since they are below 46.7% in all the items proposed. Likewise, starting with item 4c, the percentage of incorrect answers is above 51.7%, where the most frequent arguments are that the difficulties the children face is not having acquired the prior knowledge needed to solve the task or not understanding the problem statement. In particular, the pre-service teachers exhibit limitations in terms of the knowledge of content and students.

Items 2b and 3b, which consider the potential difficulties children might face when solving tasks involving establishing relationships based on attribute recognition, yielded a slightly higher percentage of correct answers compared to the other items. In the case of item 2b, the arguments presented made it possible to determine that 33.3% of the teachers noted that the possible difficulties have to do with identifying the category of each set of elements and establishing membership relationships with the elements.

Meanwhile, an analysis of the arguments given in item 3b shows that 46.7% of teachers determined that the potential difficulties are related to recognizing negative

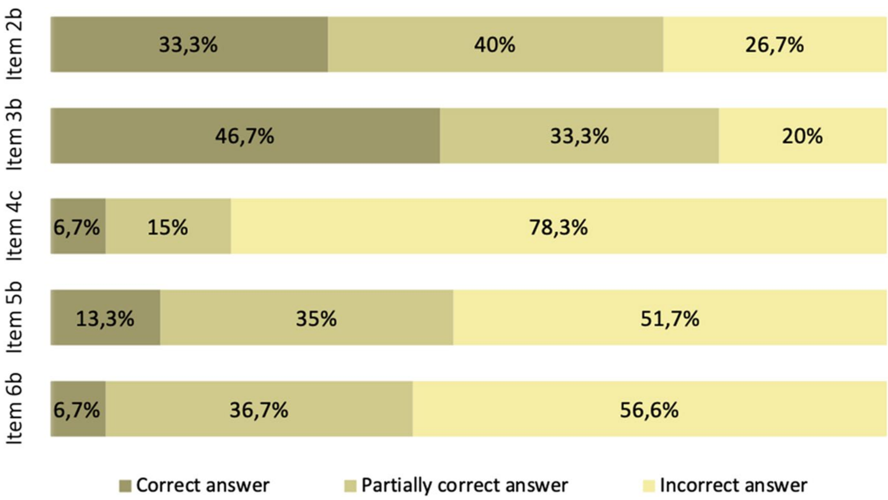


Fig. 5 Composition of the different types of answers for the knowledge of content and students by degree of correctness

attributes and discriminating the shapes, color and size of the elements proposed in the cards, for example, “children might not understand the positive and negative labels of the cards” (pre-service teacher 11). In both items, 2b and 3b, the arguments present in the partially correct answers mention that the difficulties are centered around experiencing the elements proposed in the teaching situations and their characteristics in general.

Items 4c and 5b analyze the difficulties with serialization tasks with repetition patterns. The small percentage of right answers for item 4c (6.7%) provides evidence that links the difficulties with the systematic reproduction of the pattern when extending the AB type series. 15% were of the view that the difficulties may be related to an inadequate understanding of the pattern. The remaining 78.3% failed to identify possible difficulties in the development of the task.

A similar situation is evident in the arguments presented for item 5b, where only 13.3% of the right answers propose that the difficulties involve the extension of the repetition pattern, especially in series P4, P5 and P6, which do not end in a complete repetition unit, for example, “children do not continue the pattern correctly since some series are unfinished, such as P4, P5 and P6” (pre-service teacher 44). A large percentage of the pre-service teachers failed to identify difficulties relevant to the object of study (51.7%), and the remaining 35% believed that the difficulties are related only to work with repetition patterns.

Finally, only 6.7% of the answers given for item 6b, which involves the description of qualitative change, were correct. The arguments given in these answers posit that the possible difficulties are related to the understanding of the logical operator and the representation of qualitative changes. 36.7% of the answers stated, at a very general level, that the difficulties are related to an understanding of change, “they are unable to imagine the qualitative change that is made to the object” (pre-service teacher 15). The remaining 56.6% failed to identify possible difficulties for the object of study.

Knowledge of Content and Teaching

To evaluate the knowledge of content and teaching, we focused on the use of different methods and procedures that can be used to provide instruction.

The questions asked that allow us to analyze this knowledge were: What teaching strategies would you use to help children who had problems solving the task? What resource would you use to help children solve this type of task? Explain how you would use it and justify your choice. These questions are asked in items 2c, 3c, 4d, 5c and 6c of the questionnaire. Figure 6 shows the degree of correctness of the answers provided for these items.

The results show that the pre-service teachers have a low level of knowledge of content and teaching, since the percentage of correct answers was below 40% for all the items. We did, however, see a high percentage of partially correct answers in which the teachers mention teaching strategies to address the instruction that are not entirely conclusive, as well as teaching resources; however, they do not go into detail as to how and why they would implement them in the classroom.

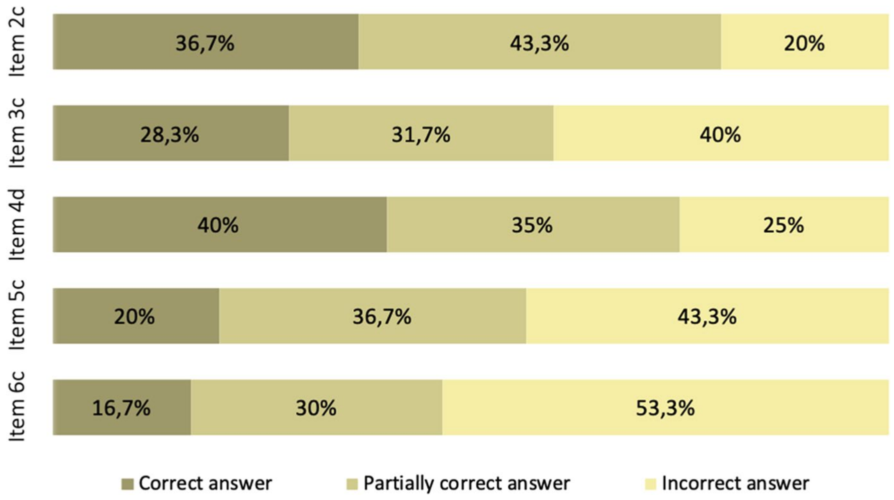


Fig. 6 Composition of the different types of answers for the knowledge of content and teaching by degree of correctness

The arguments given in the answers to item 2c, involving the recognition of attributes to establish relationships, reveal that teachers have problems establishing strategies that allow children to respond correctly to the task. 36.7% managed to describe possible teaching strategies as per the instructions in the task. Most of them suggested approaching the task experientially through manipulatives (fruits and toys) and asking questions of the type, What can we do with these objects? What characteristics do they have? in order to establish similarities and differences between the elements in each set of objects, an example of an answer is "I would present the objects physically or through cards, so the children can manipulate them and distinguish their characteristics, and then I would ask guided questions such as: What are these objects like? Can they all be eaten?" (pre-service teacher 22). A high percentage (43.3%) mentioned strategies on a very general level, but these are inconclusive; specifically, they mention working with concrete materials, providing examples through different objects, without specifying. The remaining 20% provided inadequate or meaningless strategies and resources.

Regarding item 3c, an analysis of the answers revealed that only 28.3% of the participants proposed another teaching resource to address the recognition of attributes to establish relationships, explaining and justifying their choice. They mainly proposed working with manipulatives, such as Dienes blocks, attribute blocks and others, undertaking the task initially without considering the negation of attributes. As in the previous item, a considerable percentage of responses (31.7%) were limited to mentioning only one resource, but did not detail how to use it, for example, pre-service teacher 7 points out that "I would do the activity with the Dienes blocks and cards that indicated attributes". The remaining 40% did not mention any alternative resources.

Continuing on, an analysis of the arguments given in item 4d made it possible to show that 40% of the participants proposed and justified appropriate teaching strategies to correct the error presented in the AB series with repetition patterns. They included comparison with another term-to-term series, followed by orally expressing the terms of the sequence according to the established pattern, for example, "I would ask the girl to rebuild the tower with cubes of other colors, then compare the elements in the series to see if they match. Then, I would ask the girl to describe both series aloud again" (pre-service teacher 35). 35% mentioned possible strategies, such as asking leading questions and using manipulatives to continue the sequence, without giving more details about their choice, while 25% offered strategies that do not adequately address the mistake.

In item 5c, the arguments presented showed that only 20% of the participants provided adequate strategies to help children who had problems extending the series proposed. Notable among them was proposing series in increasing order of difficulty, using other manipulatives such as Multilink cubes or colored cards, and expressing the series orally. By contrast, a large percentage (43.3%) gave meaningless strategies to deal with the task and 36.7% offered inconclusive strategies.

Finally, item 6c, which focused on proposing teaching strategies to address the ideas of change, returned a low percentage of right answers (16.7%). They proposed the use of concrete materials, mainly mentioning the machine for changing qualities, followed by describing changes using everyday situations (night and day – a tree in autumn and spring). 30% proposed strategies like the ones mentioned above in a very general way, as well as others such as posing direct questions, using examples to describe qualitative changes, without detailing what specific materials, questions and examples would be appropriate. Finally, a high percentage of answers (53.3%) proposed inadequate strategies to address the study of change.

Knowledge of the Curriculum

The set of items focused on evaluating the knowledge of the curriculum delved into central aspects of the early childhood education curriculum that pertain to a certain teaching situation.

To analyze their knowledge of the curriculum, they were asked questions of the type: considering the preschool curriculum, what could be the goal of the task? These questions are set out in items 1b, 3d, 5d and 6d. To answer the question, the teachers were expected to bring to bear their knowledge of the contents proposed in the curriculum and its intended purpose.

The degree of correctness of the answers given for these items is shown in Fig. 7.

The results reveal that the teachers have poor knowledge of the curriculum, since they exhibited difficulties interpreting the intentionality of teaching situations and linking them clearly with the guidelines set out in the curriculum. The percentage of correct answers to the proposed items was below 38.3%.

An analysis of the responses to item 1b shows that only 38.3% of the participants were able to recognize that the purpose of the task was to establish different relationships (classification, ordering) based on recognizing two or more

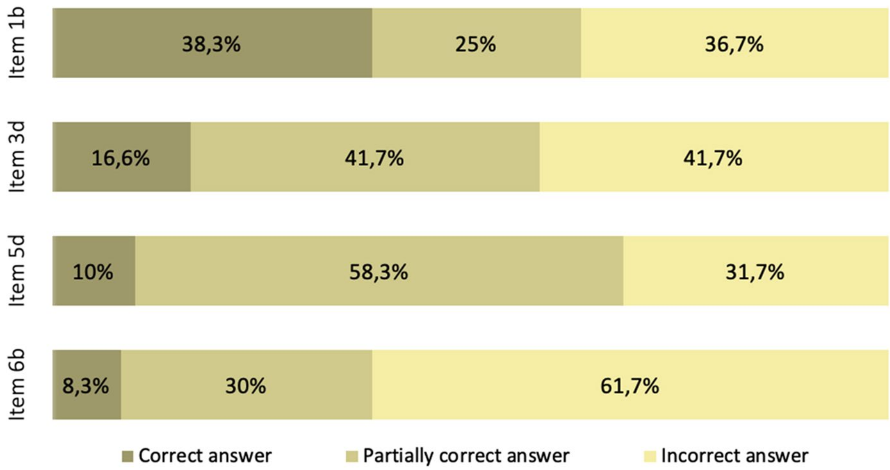


Fig. 7 Composition of the different types of answers for the knowledge of curriculum by degree of correctness

attributes at once (shape, color, size, and others), for example, "the goal of the task is to classify and sort objects based on the identification and recognition of their sensory characteristics, such as shape, color and size" (pre-service teacher 34). 25% of the participants stated very generically that the goal of the task involved establishing relationships, for example "the goal of the task is to classify certain objects in the classroom correctly" (pre-service teacher 51), while a large percentage (36.7%) failed to determine the goal of the task.

Regarding item 3d, an analysis of the answers showed that a small percentage (16.6%) correctly identified the goal of the task, which was to positively or negatively recognize attributes based on working with cards. A high percentage (41.7%) was able to identify the recognition of attributes as the goal of the task, for example, "the goal of the task is to relate attributes" (pre-service teacher 37). Similarly, 41.7% set a goal that does not correspond to the description of the task.

An analysis of the arguments given for item 5d showed that a low percentage of pre-service teachers (10%) correctly recognized that the goal of the task was to expand sequences with AB, ABC and ABB repetition patterns. A considerable percentage (58.3%) mentioned, very generally, that the goal involved working with patterns, for example "make seriations following a pattern" (pre-service teacher 56), while 31.7% failed to identify a goal.

Finally, the arguments presented for item 6d showed that only 8.3% responded correctly, by stating that the goal of the task had to do with introducing logical operators and expressing qualitative changes. Conversely, a high percentage (61.7%) failed to propose a goal for the task, while 30% only recognized that the goal of the task was related to establishing changes.

Final Considerations

In this study, we analyzed partial aspects of the mathematical knowledge of 60 pre-service early childhood education teachers to teach early algebra, from the perspective of the MKT model (Ball et al., 2008). We did so by administering the MKT-early algebra questionnaire (3–6) proposed by Pincheira and Alsina (2022a) and analyzing the answers given by the pre-service teachers to the different questions that comprise it.

As Mosvold et al. (2011) suggest, analyzing the mathematical knowledge for teaching of early childhood education teachers requires formulating teaching situations specific to this stage of schooling. By adhering to these guidelines through the questionnaire, the pre-service teachers were given tasks and situations involving manipulation and games typical of early childhood education that allow children to experience early algebraic ideas.

The study delved into the domains and subdomains of the MKT model, showing that the mathematical knowledge of early algebra of pre-service early childhood education teachers is insufficient, since the average percentage of correct answers does not exceed 26.6%.

As concerns the domain of knowledge of the subject, the interpretation of the results indicates that the subdomain of common knowledge that is required to be implemented to solve serialization tasks with repetition patterns obtained better scores compared to the other subdomains evaluated. As in the study by Cabral et al. (2021), most of the participants successfully identified the structure of the sequences; however, certain problems are observed when the sequence does not end in a complete repeat unit. This last finding agrees with the results obtained by Noviyanti and Suryadi (2019), who showed that early childhood education teachers exhibit some limitations in understanding the basic knowledge associated with working with series.

Regarding the subdomain of specialized content knowledge, we identified a lack of ability in pre-service teachers to adequately identify the mathematical content that is involved in an early algebra task, especially in tasks with repetition patterns and the study of change. Similar studies in this same area of knowledge (e.g., Bair & Rich, 2011; Pincheira & Alsina, 2022b) have concluded that early childhood education teachers have problems establishing connections between different representations of a sequence and identifying theoretical concepts in pattern tasks. Likewise, the knowledge of the mathematical horizon is one of the most complex subdomains for pre-service early childhood education teachers, given their problems establishing links between different types of mathematical relationships, such as classification and grouping, in addition to other topics in the curriculum.

We also identified considerable limitations in the domain of pedagogical knowledge of the content. In the case of the subdomain of knowledge of content and students, the pre-service teachers have problems anticipating children's thinking in relation to errors or difficulties they might have when doing a certain algebraic task. This coincides, for example, with the results obtained by Cabral et al. (2021), who stated that teachers exhibit limitations interpreting and perceiving the algebraic thinking of early childhood education children, especially in tasks involving repetitive sequences.

Regarding the subdomain of knowledge of content and teaching, teachers have insufficient knowledge to propose teaching strategies and resources that allow children to overcome their difficulties with algebraic tasks. Pincheira and Alsina (2022b) reached a similar conclusion regarding knowledge of content and teaching, noting that pre-service early childhood education teachers exhibit difficulties selecting sequences of tasks that can be used to acquire or reinforce algebraic knowledge, as well as game-based teaching strategies to motivate the development of early algebra tasks.

Finally, their knowledge of the curriculum is also insufficient, since most of the teachers could not identify the purpose of the tasks proposed in the questionnaire and relate them to the purpose pursued by the early childhood education curriculum. Perhaps the answer to this complexity is that the preschool curriculum in use in Spain explicitly considers guidance related to the representation of attributes of elements and collections, establishing grouping, classification, order and quantification relationships, and omitting work with patterns and the description of qualitative and quantitative changes (Alsina, 2022b).

From this perspective, we assume that the results obtained are subject to both the mathematical training received by participants in previous studies, as well as to the academic training they have received as pre-service teachers. In this context, the research makes it possible to identify the elements of mathematical knowledge that require further study in order to promote the development of early algebraic thinking in early childhood education; however, it is necessary to redirect the training of these pre-service teachers so that they develop mathematical knowledge that is suitable for teaching early algebra. According to Branco and Ponte (2014), teachers must know algebra and what is required to teach it in the early years of schooling in order to mobilize their didactic knowledge and their professional identity in their future practice. Against this backdrop, the situation regarding the mathematical knowledge of pre-service early childhood education teachers for teaching early algebra is worrisome and calls for reflection on what aspects to consider when training teachers to promote the development of the mathematical knowledge needed to teach early algebra in early childhood education.

We need to offer training programs to early childhood education teachers that allow them to develop this mathematical knowledge based on the following areas of action: (a) subjects focused on the study of early algebra as a mathematical object, to analyze the specific aspects of this content block and how to teach it; and (b) provide tools that allow teachers to reflect on their own teaching practice and their discursive interaction in the early childhood education classroom. Consequently, the effective incorporation of early algebra in early childhood education requires transforming teacher training to ensure it can respond to the new challenges involved in teaching this content standard. Considering the general pedagogical guidelines for training mathematics teachers (Lewis, 2016), and specifically for teaching early algebra in early childhood education (Alsina, 2022a), it would be advisable to focus the training, for example, on the analysis and discussion of real classroom situations by analyzing and reflecting on videos, to provide insights to pre-service teachers and thereby strengthen the development of the mathematical knowledge they can employ. Another element that could be considered in the training of early childhood education teachers to promote the effective teaching of early algebra is the "lesson study" (Isoda et al., 2007) as a method to develop, reflect and critically evaluate their own teaching practice, allowing them to gradually grow their mathematical knowledge and enhance various pedagogical strategies as they transfer their learning in the classroom lessons. According to Isoda et al. (2007), class study fosters collaboration and teamwork among teachers, helps improve the quality of teaching and learning of students, allows teachers to develop skills and knowledge for teaching mathematics, and promotes continuous reflection and learning of teaching practice.

Implementing lesson study requires considering the work cycle proposed by Lewis (2009), which involves bringing together pre-service teachers in small groups such that: they jointly plan a class, in this case, considering the proposal of early algebra; then, one of the members of the group implements the session in the classroom, while the rest of the members observe and; finally, together, they analyze and reflect on the implementation of the session. The analysis of the information should focus on the answers and productions (written and oral) made by the pre-service teachers over the course of the session in order to consider the reasons for their decision-making, and the implications for the ideal teaching of early algebra.

It would also be advisable to further analyze the design, implementation and redesign of early algebra tasks focused on the ability to make and express generalizations (Kaput, 2008), in order to determine in depth, the mathematical activity that children must bring to bear in their early childhood education in order to identify the solution process for an early algebra task.

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Data Availability Data generated or analysed during this study are available from the authors on request.

Declarations

Declarations of originality We declare that the manuscript and its contents is original and unpublished. It has not been submitted to other journals, is not under review for publication, and has not been presented at research conferences.

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Competing Interest Authors declare no competing interest.

Ethical approval Informed consent was obtained from all subjects involved in the study.

References

Alsina, Á. (2022a). *Itinerarios didácticos para la enseñanza de las matemáticas (3-6 años)* [Teaching routes for mathematics lessons (3-6 years)]. Editorial Graó.

Alsina, Á. (2022b). Los contenidos matemáticos en el currículo de educación infantil: Contrastando la legislación educativa española con la investigación en educación matemáticainfantil [The mathematics content in the infant education curriculum: Contrasting Spanish educational legislation with research into infant mathematics education]. *Épsilon – Revista de Educación Matemática, 111*, 67–89.

Australian Curriculum, Assessment and Reporting Authority [ACARA] (2022). Mathematics curriculum. Retrieved January 20, 2023, from https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/?layout=1

Ayala-Altamirano, C., & Molina, M. (2020). Meanings attributed to letters in functional contexts by primary school students. *International Journal of Science and Mathematics Education, 18*, 1271–1291. https://doi.org/10.1007/s10763-021-10218-6

Bair, S. L., & Rich, B. S. (2011). Characterizing the development of specialized mathematical content knowledge for teaching in algebraic reasoning and number theory. *Mathematical Thinking and Learning, 13*(4), 292–321. https://doi.org/10.1080/10986065.2011.608345

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Branco, N., & Ponte, J. (2014). Articulação entre pedagogia e conteúdo na formação inicial de professores dos primeiros anos: Uma experiência em Álgebra [Articulation between pedagogy and initial training content of first-year teachers: An experience in algebra]. In J. Ponte (Ed.), *Práticas profissionais dos professores de matemática* (pp. 379–408). Instituto de Educação da Universidade de Lisboa.

Cabral, J., Oliveira, H., & Mendes, F. (2021). Preservice teachers' mathematical knowledge about repeating patterns and their ability to notice preschoolers algebraic thinking. *Acta Scientiae (Canoas), 23*(6), 30–59. https://doi.org/10.17648/acta.scientiae.6302

Corbin, J. M., & Strauss, A. L. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (3rd ed.). Sage Publishers.

Creswell, J. W. (2014). *Research design: Qualitative, quantitative, and mixed methods approaches* (4th ed.). Sage Publications.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372–400.

Isoda, M., Stephens, M., Ohara, Y., y, & Miyakawa, T. (Eds.). (2007). *Japanese lesson study in mathematics: Its impact, diversity and potential for educational improvement.* World Scientific.

Kaput, J. (2000). *Transforming algebra from an engine of inequity to an engine of mathematical power by algebrafying the K-12 curriculum*. National Center for Improving Student Learning and Achievement in Mathematics and Science.

Kaput, J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5–17). Lawrence Erlbaum. https://doi.org/10.4324/9781315097435-2

Kieran, C., Pang, J., Schifter, D., & Ng, S. F. (2016). *Early algebra: Research into its nature, its learning, its teaching*. Springer.
Krippendorff, K. (2013). *Content analysis. An introduction to its methodology* (3rd ed.). Sage Publications.
Lane, K. L., Oakes, W. P., Powers, L., Diebold, T., Germer, K., Common, E. A., & Brunsting, N. (2015). Improving teachers' knowledge of functional assessment-based interventions: Outcomes of a professional development series. *Education and Treatment of Children, 38*(1), 93–120. https://doi.org/10.1353/etc.2015.0001
Lenz, D. (2022). The role of variables in relational thinking: an interview study with kindergarten and primary school children. *ZDM Mathematics Education,* 1181–1197. https://doi.org/10.1007/s11858-022-01419-6
Lewis, C. (2009). What is the nature of knowledge development in lesson study? *Educational Action Research, 17*(1), 95–110. https://doi.org/10.1080/09650790802667477
Lewis, C. (2016). How does lesson study improve mathematics instruction? *ZDM Mathematics Education, 48*, 571–580. https://doi.org/10.1007/s11858-016-0792-x
Lüken, M. M., y, & Sauzet, O. (2020). Patterning strategies in early childhood: A mixed methods study examining 3- to 5-year-old children's patterning competencies. *Mathematical Thinking and Learning, 23*(1), 28–48. https://doi.org/10.1080/10986065.2020.1719452
Ministerio de Educación y Formación Profesional [MEFP] (2022). *Real decreto 95/2022, de 1 de febrero, por el que se establece la ordenación y las enseñanzas mínimas de la educación infantil* [Royal decree 95/2022, of 1 February, establishing the order and minimum teaching levels of infant education]. MEFP.
Ministry of Education Singapore. (2013). *Nurturing early learners: A curriculum for kindergartens in Singapore: Numeracy: Volume 6*. Ministry of Education.
Mosvold, R., Bjuland, R., Fauskanger, J., & Jakobsen, A. (2011). Similar but different–investigating the use of MKT in a Norwegian kindergarten setting. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the seventh congress of the European society for research in mathematics education* (pp. 1802–1811). CERME.
National Council of Teachers of Mathematics [NCTM]. (2000). *Principles and standards for school mathematics*. NCTM.
Noviyanti, M., & Suryadi, D. (2019). Basic mathematics knowledge of early childhood teachers. *Journal of Engineering Science and Technology, 1*, 19–27.
Pincheira, N., & Alsina, Á. (2021a). Hacia una caracterización del álgebra temprana a partir del análisis de los currículos contemporáneos de educación infantil y primaria [Gives a characterization of temporal numeracy from the analysis of contemporary primary and elementary school curricula]. *Revista Educación Matemática, 33*(1), 153–180. https://doi.org/10.24844/EM3301.06
Pincheira, N., & Alsina, Á. (2021b). Teachers' mathematics knowledge for teaching early algebra: A systematic review from the MKT perspective. *Mathematics, 9*, 2590. https://doi.org/10.3390/math9202590
Pincheira, N., & Alsina, Á. (2022a). Evaluación del conocimiento para enseñar álgebra temprana durante la formación inicial del profesorado de educación infantil [Assessment of knowledge for teaching advanced algebra during the initial training of the teacher of infant education]. *Revista deInvestigación en Educación, 20*(2), 154–171. https://doi.org/10.35869/reined.v20i2.4222
Pincheira, N., & Alsina, Á. (2022b). Mathematical knowledge of pre-service early childhood and primary education teachers: An approach based on the design of tasks involvingpatterns. *Australian Journal of Teacher Education, 47*(8), 50–69. https://doi.org/10.14221/ajte.2022v47n8.4
Pinto, E., Cañadas, M. C., & Moreno, A. (2022). Functional relationships evidenced and representations used by third graders within a functional approach to early algebra. *International Journal of Science and Mathematics Education, 20*, 1183–1202. https://doi.org/10.1007/s10763-021-10183-0
Rittle-Johnson, B., Fyfe, E. R., Loehr, A. M., & Miller, M. R. (2015). Beyond numeracy in preschool: Adding patterns to the equation. *Early Childhood Research Quarterly, 31*, 101–112. https://doi.org/10.1016/j.ecresq.2015.01.005
Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14. https://doi.org/10.3102/0013189X015002004
Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1–22. https://doi.org/10.17763/haer.57.1.j463w79r56455411
Torres, M. D., Moreno, A., Vergel, R., & Cañadas, M. C. (2024). The evolution from "I think it plus three" towards "I think it is always plus three." Transition from arithmetic generalization to

algebraic generalization. *International Journal of Science and Mathematics Education*. https://doi.org/10.1007/s10763-023-10414-6

Walkoe, J., Walton, M., & Levin, M. (2022). Supporting teacher noticing of moments of algebraic potential. *Journal of Educational Research in Mathematics, 32*(3), 271–286. https://doi.org/10.29275/jerm.2022.32.3.271

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