

# Electoral Inequity

Nicolas Boccard\*

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## Abstract

Ideally, a representative democracy awards a *genuine* vote to each adult. We study this issue in competitive democracies with an election model combining district apportionment and proportional representation (PR). Four classic seat allocation rules, including [d'Hondt \(1882\)](#), are reframed as Dutch auctions, allowing important properties to be easily derived. The pros and cons of these methods are stated in terms of economic inequality; [Sainte Laguë \(1910\)](#)'s is shown to best carry the *genuine* vote ideal, both for elections and for apportionment. We next expound the interplay between these two components in generating an inequitable treatment of voters and develop the scale-free index of inequity best fitted to their concern. We apply it to 40 countries for the apportionment of electoral districts. Lastly, we compute the same inequity index for recent parliamentary elections in 80 countries, finding that the majority system mistreats electors, thus putting a “price” on government stability.

Keywords: Election; Apportionment; Divisor method; Equity; d'Hondt; Sainte Laguë; Disproportionality

JEL codes : D72, D63, K16, C43

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\*Econ. Dep., U. of Girona, Spain (nicolas.boccard@udg.edu). Financial support from *Generalitat de Catalunya* (AGAUR SGR 1360) and *Ministerio de Ciencia, Innovación y Universidades* (PID2019-106642GB-I00). [△ code](#) ☺ [A. Elbakyan](#).

# 1 Introduction

To radical political activists [Wilkes \(1776\)](#) and [Mirabeau \(1789\)](#), an ideal parliament arising from elections should be a “spitting image” of the nation,<sup>1</sup> a view properly synthesized by [Mill \(1861\)](#) as “the pure idea of democracy is the government of the whole people by the whole people, *equally* represented”, so that “a majority of the electors would always have a majority of the representatives; but a *minority* of the electors would always have a *minority* of the representatives” (§7, emphasis added).<sup>2</sup> The now ubiquitous *universal franchise* in competitive democracies has achieved the first part of this proportional representation (PR) ideal<sup>3</sup> but as far as the delicate balance highlighted in the second part is concerned, there remains many instances where sizable and well delineated minority opinions or groups are left without political representation, sometimes at the hand of an abusive majority employing dilution strategies (aka debasement).

Although constitutions profess the political equality of citizens with respect to legislative representation, they only emit proportionality *guidelines* for apportionment and elections, leaving the task of implementation to governments and oversight to courts. This arises from the complexity of obtaining and compressing millions of opinions into a small number of elected candidates that shall govern and legislate. Whereas the majority is easily identified and served its due, every other grouping is but a minority. Whichever contour it assumes (e.g., geographical, religious), its fair representation will be a subjective matter and its implementation a fierce political struggle. Political philosophers [Grotius \(1625\)](#), [Locke \(1695\)](#) and [Rousseau \(1755\)](#) thus advocated for *majoritarianism* whereby the majority faction in parliament holds an absolute right to govern, “at will”, unencumbered by opposition.<sup>4</sup> In hindsight, [Popper \(1988\)](#) adds a further quality to majoritarianism, the ability to swiftly dismiss a “bad” government.<sup>5</sup> This framework has thus endured in many countries, only surrendering a slice of PR to vocal minorities without however appeasing all their qualms.

In recent decades, buoyed by growing individualism and liberalism, aggrieved citizens of advanced economies have successfully challenged electoral laws in constitutional courts on the ground that every vote should weight equally upon the election result if political rights are to be truly equal among citizens. In several cases discussed in §3.1, landmark rulings have mandated governments making sure their citizens were endowed with what has been alternatively termed, an *effective*,

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<sup>1</sup> [Wilkes](#) dreams “an English Parliament to speak the free unbiassed sense of the body of the English people” (p. 1295), while for [Mirabeau](#), “les états sont pour la nation ce qu’est une carte réduite pour son étendue physique; soit en partie, soit en grand, la copie doit toujours avoir les mêmes proportions que l’original” (p. 7).

<sup>2</sup>Sadly, this most progressive vision was limited to the homeland and was not meant to apply to British imperial dominions such as Ireland or India as recalled by [Kawana \(2010\)](#) and [Tunick \(2006\)](#) respectively.

<sup>3</sup>cf. [Brighthouse and Fleurbaey \(2010\)](#) for a modern defense.

<sup>4</sup>cf. [Klemperer \(2019\)](#) on Britain. Pushing the argument to the limit, authoritarian regimes claim to directly represent the people and avoid submitting themselves to “messy” competitive elections.

<sup>5</sup>Going even further, the UK or Australian party in power can swiftly dismiss its entire government with a leadership contest vote from a majority of its MPs.

*genuine* or *worthy* vote. Our aim here is to measure *electoral worth*, theoretically and then empirically, across many countries over recent decades. For that task, we conceptualize *electoral inequity* as the statistical dispersion of individual electoral worthiness.

The literature around the issue of electoral equality (or equity) has concentrated on representatives (MPs) and territories rather than citizens (cf. classics by [Duverger \(1951\)](#), [Rae \(1967\)](#), [Monroe \(1994\)](#), [Lijphart \(1994\)](#), [Samuels and Snyder \(2001\)](#) or more mathematical studies by [Taagepera and Grofman \(2003\)](#), [Schuster et al. \(2003\)](#), [Karpov \(2008\)](#)). We summarize the methods involved §2.5, noting they mostly abstain from eliciting a moral criteria and then finding the optimal rule to satisfy it (within the whole-numbers limit). In the next section, we build the notion of a “worthy vote” from an intuitive approach. We then synthetically model apportionment and election so as to contain both pure majoritarianism and pure proportionality, as well as some classical PR methods whose equity properties are derived to allow a clearcut comparison. Sainte Laguë will be shown to be most equitable. The appendix offers two detailed examples of inequity computation. The third empirical section assesses an almost exhaustive sample of competitive democracies, firstly for the latest apportionment of representatives among states/regions/provinces, highlighting the deeply unequal treatment of residents according to non-constitutional details. Next, we collect the results of some 500 general elections held since 1990 to compute the effective electoral influence of voters.

Unsurprisingly, we confirm that majoritarianism fails by a large margin the ideal of “genuine voter equality”. Importantly, we account for all the citizens excluded from representation by the endogenous inner workings of their political system and the exogenous geographical distribution of political preferences. As a mean to compare countries and epochs, we compute an equivalent share of “disenfranchised” people (among the population going to the voting booth) which puts a price upon the political stability that pure majoritarianism is supposed to bring about.<sup>6</sup>

## 2 Theory

We first develop the notion of a “worthy vote”, then proceed to give an intuitive economic meaning to the (sometimes confusing) popular rounding methods used in elections and apportionment, down to sensible proofs. Next, we associate these methods with clearcut social equity objectives, ordered on the basis of recent constitutional rulings. Our formal results mostly originate with [Sainte Laguë \(1910\)](#) and [Balinski and Young \(1982\)](#) (cf. also [Pukelsheim \(2014\)](#)).

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<sup>6</sup>Recall that when small parties are needed to reach majority in parliament, they hold-up the government, asking for ever large favors. Such pork-barrel politics then impede ambitious country wide reforms and paralyze governments.

## 2.1 What is Electoral Worth ?

In a binary vote aiming to discriminate among two candidates or policy proposals, it seems fairly obvious that an elector casts a *worthy* vote whenever her choice wins. Since electors are political equals, we should, as external observers, allocate the same amount of worth to each “winner”, say one point. The majority rule then maximizes aggregate worth, even though a large minority still gets a zero worth. Observe now that if such votes are repeated weekly over a variety of issues such that preferences are widely distributed across the electorate, then everybody ends the year with a similar amount of points; individual worth will be more or less equalized across the electorate, as constitutionally mandated. Nevertheless, if a group of like minded people is always on the losing side, they finish with no worth at all and we may be tempted to give them redress. But, should we change the decision rule away from majority in order to compensate them, an even greater number of people would get zero worth.<sup>7</sup> Now, since electors are equals in rights, this would be a collective loss or error. We may thus conclude that the majority rule brings about the greatest total worth while minimizing the share of zero-worth voters.

In a general election where dozens of parties/candidates vie for millions of votes, electoral worth is naturally gauged by the success of the party endorsed by each voter, which in a first instance is the number of seats they win; it will vary from zero for supporters of fringe parties to a maximum for the premier party of the country. Since many voters endorse each party, success points must be shared among all of them; the correct measure of individual worth is then the seats-to-votes ratio, ranging across the electorate from zero to some maximum. Under ideal political equality (and full participation), every individual worth would be equal to the ratio of parliament seats to aggregate electorate. In actual elections, worth will always diverge from this ideal. J.S. Mill’s ideal representative democracy thus becomes ambivalent due to the tension between the dual objectives of satisfying majority and minority; indeed, there is simply not enough seats to content all minorities, especially the smaller ones without harming the majority.

It is nevertheless remains clear that an election with concentrated worth is more equitable to the electorate than one where a single party and its supporters achieve maximal worth and everyone else zero (e.g., as in a dictatorship running phony elections). We may thus claim that electoral *inequity* is naturally gauged by the dispersion of worth across the electorate. The standard measure for such problem is well known, it is the statistical concept of “standard deviation” and better still, the scale free “coefficient of variation”.<sup>8</sup> Because an apportionment of parliament among territories is, from a mathematical point of view, identical to an election, it is also amenable to the same equity measurement. Summarizing,

**Definition 1** *Electoral inequity is the dispersion of individual worth around an ideal PR national*

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<sup>7</sup>An argument in favor of such a scheme of positive discrimination is that the loss of worth may be randomly distributed and thus matter very little to the majority members.

<sup>8</sup>Alternative measures of dispersion are discussed in §2.5.

*mean. It applies to apportionment (resident) and election (elector).*

To minimizing inequity, one should combine a single nationwide district and the Sainte Laguë seat awarding rule (as shall be shown later). Practically, the first feature is rare while the second is far from ubiquitous while the combination isn't used anywhere yet. This oddity may be explained by the fact that early democracies were majoritarian, building on single-seat districts and plurality (aka first-past-the-post). Even though most have since adopted multi-member constituencies based on historical territories and a PR allocation rule (e.g., d'Hondt), they remain far from pure proportionality. Furthermore, the judicial fight for true political equality (as professed in constitutions) is a novel one whose impact is only slowly sipping since it is systematically opposed by those parties or territories, in the case of apportionment, who stand to lose and which, by construction, have an excessive representation and likely a blocking minority. As the judicial examples in 3.1 show, only the superior authority of a constitutional court is able to break the deadlock.

Concluding, we must recall that even under pure proportionality, some inequity remains for everyone in the nation. Now, as the best rule is still imperfect, every seat allocation rule has a claim to offer something valuable and this is probably why debates rage on that matter.

## 2.2 Electoral Model

In a representative democracy, the electoral system regularly gathers millions of votes and returns a parliamentary seat distribution among political parties (or individual candidates).<sup>9</sup> Two stages are involved, *apportionment* and *election*. The first is a required partition of the national territory into a fixed number of voting districts (aka constituencies), each returning a fixed number of parliamentary seats. We consider 3 basic schemes whose naming mirrors their influence upon the election. Under *pure proportionality* (aka “at-large” election), a single district elects all MPs at once while at the *majoritarian* polar opposite, there are as many single-seat districts as seats in parliament, each awarded by plurality.<sup>10</sup> Outside these extremes, one speaks of PR because seats are awarded to constituencies roughly in proportion to their population (cf. §3.2 for details and analysis).

The *election* stage, in turn, is a complex process with many important facets; we focus on the formula turning party votes into an allocation of parliamentary seats. We assume universal franchise and some adherence to proportionality as mandated by constitutions everywhere; we consider 4 prevalent implementations conveniently labeled *greatest remainder*, *d'Hondt*, *Sainte Laguë* and *American*,<sup>11</sup> yielding identical results in majoritarian elections and nearly so for “at-large”

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<sup>9</sup>Many countries have a bicameral system with an upper house (senate) representing territories. Since the latter's political power is often weak, we concentrate on the lower house representing people, hereafter parliament.

<sup>10</sup>We include here the “run-off” system, limiting the second round to the 2 most voted candidates, so as to make the winner elected by a majority of valid votes. We exclude “block-voting” whereby all seats are awarded to the most voted faction since this electoral modality has mostly disappeared.

<sup>11</sup>They were originally discovered by American scholars Hamilton, Jefferson, Webster and Hill.

ones; they start differing solely for intermediate designs when each constituency returns somewhere between 3 and 12 seats.

Formally,  $H$  parliamentary seats (e.g., 500) are firstly apportioned between  $m$  districts or constituencies as a vector  $\mathbf{h} = (h_d)_{d \leq m} \in \mathbb{N}^m$  such that  $H = \sum_{d \leq m} h_d$ ; this is often done in proportion to population but other criteria may be used as well as we detail in the empirical section §3.2. For the election of the  $h$  representatives of a particular district, the  $n$  contending parties receive valid votes  $\mathbf{v} = (v_i)_{i \leq n} \in \mathbb{N}^n$ . Once a feasible allocation  $\mathbf{a} = (a_i)_{i \leq n} \in \mathbb{N}^n$  such that  $h = \sum_{i \leq n} a_i$  has been determined by the current electoral rule, we may compute for each party its *effective quotient* or “price”  $p_i = \frac{v_i}{a_i}$  counting how many votes go into each seat; it is infinite if the party is unsuccessful. We shall however devote ourselves to *worth*  $w_i = \frac{a_i}{v_i}$  which counts how many MPs a party gets per million votes (aka. “seats-to-votes ratio” in the literature); it is zero for unsuccessful (fringe) parties. Crucially, every elector derives an ex-post *worth*, that obtained by the party she voted for; it is thus an individual measure of representation as well. In empirical applications, we shall only count the valid votes received by parties, thus excluding blanks, nulls and abstentions since these categories pertain to the related but distinct problem of participation in democratic elections. Clearly, equity indices  $p_i$  and  $w_i$  may be computed at the national level by summing party votes and seats across all constituencies. We may now say that party # $i$  is *advantaged* over party # $j$  if either  $p_i < p_j$  or  $w_i > w_j$  i.e., the former needs less votes than the latter to gain a seat or equivalently, that  $i$ -voters are more decisive than  $j$ -voters.<sup>12</sup>

Ideal proportional representation uses the vote aggregate  $v_0 \equiv \sum_{i \leq n} v_i$  to build the *natural quotient*  $p^* = \frac{v_0}{h}$  counting how many votes go into one parliamentary seat (e.g., 30 000) while its inverse, the mean worth  $w^* = \frac{h}{v_0}$ , counts how many MPs are returned per million (valid) votes. The *fair share* of party # $i$ 's is then  $a_i^* = \frac{v_i}{p^*} = \frac{v_i}{v_0} h$ , measuring how many decimal seats it deserves under “pure PR”. If it was feasible, every MP would have the same quotient  $p^*$  and every voter the same worth  $w^*$ . A practical PR method is then any systematic way of rounding decimal fair shares into natural numbers (constitutions never permit sharing seats). We use the integer part  $\lfloor x \rfloor$  of a number  $x$  as well as its commercial rounding  $\langle x \rangle$ .

Whatever the allocation rule in use, no party will ever receive its exact fair share; hence, electoral inequity is bound to be present on all sides. Historically, the contention among alternative rounding methods has revolved around their perceived intuitive pros and cons; later, mathematical analysis enabled to assess their equity properties in more systematic fashion and move the debate towards their morally desirable features (or lack thereof). We shall argue that [Sainte Laguë \(1910\)](#)'s method possesses a strong ethical meaning that awards a preeminent moral position.

<sup>12</sup>Some authors in the literature use an *absolute* view whereby party  $i$  is advantaged over  $j$  if  $a_i - a_i^* > a_j - a_j^*$ .

## 2.3 Proportional Representation Rules

We introduce 4 classical rounding methods implementing proportional representation. The *largest remainder* method<sup>13</sup> builds on the natural quotient  $p^*$  to allocate a seat for every batch of  $p^*$  votes handed over by parties; it then distributes the remaining ones, in fairness, one each to the parties holding most unused votes. Economically, this two-stage *market clearing* uses individual votes as currency: the electoral commission auctions  $h$  seats at price  $p^*$  to parties endowed with votes; in the first round of exchange,  $\hat{h} = \sum_{i \leq n} \lfloor \frac{v_i}{p^*} \rfloor$  seats are sold and since  $\hat{h} < h$  is a statistical certainty, the auctioneer must lower the price further down to some  $\hat{p}$  in order to clear the secondary market of the  $h - \hat{h}$  remaining seats. As this goes on, the party leader with most remaining unused votes will be the first to raise her hand to signal her willingness to acquire one last seat. The auctioneer will keep lowering the price until enough hands are raised. Observe now that if  $k = p^* - \hat{p}$  votes are given for free to each party, the first stage demand for seats will be  $\hat{h} + (h - \hat{h}) = h$ . We may therefore conclude that the remainder method seeks the lowest natural  $k$  solving  $\sum_{i \leq n} \lfloor \frac{v_i + k}{p^*} \rfloor = h$ . The seat allocation is then  $\mathbf{a} = \left( \lfloor \frac{v_i + k}{p^*} \rfloor \right)_{i \leq n}$ . From a marketing point of view, the auctioneer hands out  $k$  “free” votes to each party to boost “sales”; this is akin to a 5\$ supermarket coupon to be applied at check-out.

The next 3 *quotient* methods are akin to *Dutch* auctions. For [d’Hondt \(1882\)](#), the auctioneer starts by quoting price  $p^*$  and then lowers it, waiting for parties to signal their willingness to buy one or more seats. This goes on until the aggregate demand equates the fixed supply of  $h$  seats, at the equilibrium price  $\hat{p}$  solving  $\sum_{i \leq n} \lfloor \frac{v_i}{\hat{p}} \rfloor = h$  yielding the seat allocation  $\mathbf{a} = \left( \lfloor \frac{v_i}{\hat{p}} \rfloor \right)_{i \leq n}$  (to be shown shortly). In this auction, the demand schedule of party  $\#i$  is the staircase function formed by the successive levels  $\frac{v_i}{k}$  for  $k \in \mathbb{N}^*$ , with each being their willingness to pay (WTP) for  $k$  seats. At the equilibrium  $\hat{p}$ , party  $\#i$  buys  $a_i$  seats i.e., her willingness to pay for those seats is at least the going price but she is not ready to acquire another one i.e.,  $\frac{v_i}{a_i} \geq \hat{p} > \frac{v_i}{a_i + 1} \Leftrightarrow a_i \leq \frac{v_i}{\hat{p}} < a_i + 1 \Rightarrow \lfloor \frac{v_i}{\hat{p}} \rfloor = a_i$ , as claimed. Computing the series of WTP ratio  $\frac{v_i}{k}$  allow an easy practical derivation of the equilibrium: it is enough to classify them by descending order and pick the first  $h$  to know exactly who gets what; notice that the last ratio is exactly  $\hat{p}$ .

The [Sainte Laguë \(1910\)](#) method is a variation solving  $\sum_{i \leq n} \langle \frac{v_i}{\hat{p}} \rangle = h$  i.e., swap lower rounding with commercial rounding. In economic language, this is akin to hand out half-a-seat to each party and continue with the Dutch auction. Indeed, with this gift, the demand schedule of party  $\#i$  is now  $\frac{v_i}{k - 1/2}$  for  $k \in \mathbb{N}^*$  and at the equilibrium  $\hat{p}$ , each party  $\#i$  buys  $a_i$  seats with the property  $\frac{v_i}{a_i - 1/2} \geq \hat{p} > \frac{v_i}{a_i + 1/2} \Leftrightarrow a_i \leq \frac{v_i}{\hat{p}} + \frac{1}{2} < a_i + 1 \Rightarrow \langle \frac{v_i}{\hat{p}} \rangle = \lfloor \frac{v_i}{\hat{p}} + \frac{1}{2} \rfloor = a_i$ . In marketing terms, the auctioneer gives each party a 50% rebate *coupon* valid for the first seat, to boost “sales”. Politically, the coupon helps smaller parties obtain their first seat as compared to the d’Hondt methods well known to favor major parties.<sup>14</sup>

<sup>13</sup>Devised by [Hamilton \(1792\)](#) for US apportionment and known in Europe as the Swiss method since 1890.

<sup>14</sup>The steeper slopes of the bid schedules of the auction participants in the Sainte Laguë version force a lower equi-

The *American* method,<sup>15</sup> employs a variable first seat discount  $\delta_k = k - \sqrt{k(k-1)}$  when acquiring  $k$  units (i.e.,  $\delta_1 = 100\%$ ,  $\delta_2 = 69\%$ ,  $\delta_3 = 55\%$  converging towards 50% for  $k > 8$ ).<sup>16</sup> Interestingly, the largest  $h$  parties obtain representation; for an apportionment, this means that every territory is represented. In summary:

**Result 1** *For an election or an apportionment, the 4 classic PR methods are descending auctions allocating a fixed number of seats among potential acquirers endowed with their votes or resident counts; they only differ by the promotional coupon offered to acquirers.*

## 2.4 Assessing Inequity

We now compare the 4 methods in terms of embedded electoral inequity. For a single-seat district ( $h = 1$ ), all award the seat to the most voted party (thus giving zero worth to voters of other parties) whereas if  $h$  is large, say, many times over the number of contending parties, the equilibrium price is so low that all meaningful parties gain MPs and none will see any difference between the 4 methods (and quarrel over which should be used). Between these extremes, [Sainte Laguë \(1910\)](#) shows that each of the 4 PR methods minimizes a global objective under the twin constraints of using naturals and summing to district size  $h$ . Given a candidate seating  $\mathbf{a} = (a_i)_{i \leq n}$  and some ideal concept, the mean squared error (MSE) is the standard statistic to assess goodness-of-fit.<sup>17</sup>

**Result 2** *Among the 4 classic PR seat allocation methods,*

- **d'Hondt** *reduces the advantage of the most favored party.*
- **Largest Remainder** *minimizes the MSE of party allocation  $\Phi = \sum_i (a_i - a_i^*)^2$  (wrt. fair shares).*
- **American** *minimizes the MSE of effective quotient  $\Psi = \sum_i a_i (p_i - p^*)^2$  (wrt. national mean).*
- **Sainte Laguë** *minimizes the MSE of voter worth  $\sigma^2 = \sum_i \frac{v_i}{v_0} (\omega_i - \omega^*)^2$  (wrt. national mean).*

Proof For *d'Hondt*, we've already shown that  $\max_j \frac{v_j}{a_j+1} \leq \hat{p} \leq \min_i \frac{v_i}{a_i} \Leftrightarrow \max_i \frac{a_i}{v_i} \leq \mu \leq \min_j \frac{a_j+1}{v_j}$  with  $\mu = \frac{1}{\hat{p}}$ . Thus, the rule at work is  $\min_{\mathbf{a}} \max_i \frac{a_i}{v_i}$  which reduces the advantage of the most favored party. With the *largest remainder* rule, the objective  $\Phi$  after the first round of seat distribution is  $\sum_i (a_i^* - \lfloor a_i^* \rfloor)^2$  i.e., the sum of squared remainders. Hence, to further shrink  $\Phi$ , we award the last seats to parties with the greatest remainder to minimize the distance between 1 and the remainder. With the *Sainte Laguë* rule, the objective is  $\sigma^2 = \sum_i \frac{v_i}{v_0} (\omega_i - \omega^*)^2 \propto \sum_i \frac{a_i^2}{v_i} - v_0 \omega^*$  by the variance decomposition formula, so we may drop the constant to find the minimum. Now, optimality asks

librium price which favors acquisition by "poorer" parties.

<sup>15</sup>This rule serves in the US to apportion seats among states for the House of Representatives in proportion to states population. [Hill \(1911\)](#) derives it as a locally PR optimum while [Sainte Laguë \(1910\)](#) obtains it as a global PR optimum.

<sup>16</sup>The demand schedule is now  $\frac{v_i}{\sqrt{k(k-1)}}$  and the auction equilibrium condition becomes  $\frac{v_i}{\sqrt{a_i(a_i+1)}} \leq p^* < \frac{v_i}{\sqrt{a_i(a_i-1)}}$ .

<sup>17</sup>Strictly speaking, each of the 3 formulas shown is a proportion of the relevant MSE.



that for any pair  $(i, j)$ , swapping a seat from  $i$  to  $j$  increases the objective i.e.,  $\frac{a_i^2}{v_i} + \frac{a_j^2}{v_j} \leq \frac{(a_i-1)^2}{v_i} + \frac{(a_j+1)^2}{v_j} \Leftrightarrow \frac{2a_i-1}{v_i} \leq \frac{2a_j+1}{v_j} \Leftrightarrow \frac{v_j}{a_j+1/2} \leq \frac{v_i}{a_i-1/2}$  which is the auction equilibrium condition. Lastly, in the *American* case, the objective, restricted to parties awarded seats, is  $\Psi = \sum_i a_i (p_i - p^*)^2 = \text{cte} + \sum_i a_i p_i^2 = \sum_i \frac{v_i^2}{a_i}$ ; at the optimum, it must be true that  $\frac{v_i^2}{a_i} + \frac{v_j^2}{a_j} \leq \frac{v_i^2}{a_i-1} + \frac{v_j^2}{a_j+1} \Leftrightarrow \frac{v_j}{\sqrt{a_j(a_j+1)}} \leq \frac{v_i}{\sqrt{a_i(a_i-1)}}$ , the auction equilibrium condition. ■

We may now reinterpret Result 2 from an ethical point of view in order to identify a “constitutional” ordering among our 4 candidate PR systems: the *d’Hondt* rule displays a marked extremism by focusing exclusively on a worst case, completely disregarding possible inequities existing among other parties (and by extension in the rest of the population that did not vote for the most advantaged party). This disqualifies it with respect to the constitutional requirement of equal voting effectiveness. Yet, being the PR rule most advantageous to large parties, it may appear as a middle ground between the demands for political stability and electoral equity; this might explain the lasting popularity of *d’Hondt* across the world.<sup>18</sup>

The *largest remainder* criteria addresses party inequity. The symmetrical treatment of parties seem attractive at first glance, but hides several drawbacks: firstly, a party splitting in two identical halves (votes and seats) reduces  $\Phi$  due to the convexity of  $x^2$  even though neither parties nor voters see their equity measures affected; obviously, forming a coalition produces the reverse phenomenon. Next,  $\Phi$  treats equally all parties so that a small party with 10% of the vote but 5% of seats would contribute 25 points to  $\Phi$  exactly as a major party with 30% of the vote but 35% of seats; there is however a key difference as the latter represent three times as many voters, meaning that the small party claim (for obtaining more seats) does not carry much moral weight.

Since members of parliament (MP) enjoy equal rights and duties, including the same right to vote, it stands to reason they should get into parliament with similar popular support as measured by their voting quotient. The dispersion of MP quotients around the national mean is naturally gauged by criteria  $\Psi$  of the *American* method where unsuccessful votes for fringe parties are ignored as the summation only gives weight to parties that gained parliamentary seats. For an apportionment such as the US house of representatives, the *American* rule seeks to equate the number of people being represented by each seating member, a fine intuitive criteria. Now, when applied to an election, there must be a vote threshold to avoid pandemonium among hundreds of micro parties; we may then say that this rule seeks equality among the insiders of the electoral game which remains unfair to voters endorsing novel emerging political sensitivities. As we shall show later on, the *American* rule possesses several of the useful properties we are about to discover.

Lastly, *Sainte Laguë* directly tackles the unequal treatment of voters: the dispersion of voter worth around the national mean is gauged by the variance  $\sigma^2$ . Since constitutions favor the equal

<sup>18</sup>As shown by Palomares and Ramírez (2003), the only divisor method guaranteeing a seat majority to the party winning a majority of the vote, under all circumstances, is *d’Hondt*.

treatment of voters over the equal treatment of representatives or parties, we may rank *Sainte Laguë*, *American* and *largest remainder* from best to acceptable in their effort to redress the inequities necessarily generated by the rounding of fair quotas (to implement PR). Germany is one of the countries that recently adopted *Sainte Laguë* in response to constitutional challenges.

## 2.5 Inequity across time and politics

In the political science literature, starting with [Duverger \(1951\)](#), electoral inequity has been known as *malapportionment* for districting and *disproportionality* for elections; there has also been a marked tendency for authors to focus on parties and provinces (states) rather than voters and citizens: [Loosemore and Hanby \(1971\)](#) gauge electoral (party) inequity with the index of distortion<sup>19</sup>  $\Omega = \frac{1}{2} \sum_i \left| \frac{v_i}{v_0} - \frac{a_i}{h} \right| \propto \sum_i |a_i^* - a_i|$ , a formula close to the largest remainder objective  $\Phi$ . Unfortunately, among two under-represented parties ( $a_i^* > a_i$ ), the  $\Omega$  index is indifferent as to how their total seats are allocated, a clear violation of [Dalton \(1920\)](#)'s transfer principle (to be discussed soon).

Adopting an econometric approach to improve upon  $\Omega$ , [Gallagher \(1991\)](#) finds the largest remainder method and upon noticing that  $h^2\Phi = \sum_i \left( \frac{v_i}{v_0} - \frac{a_i}{h} \right)^2 \leq \sum_i \left( \frac{v_i}{v_0} \right)^2 + \sum_i \left( \frac{a_i}{h} \right)^2 \leq 2$ , he proposes to use the (bounded) percentage  $\rho = h\sqrt{\Phi/2}$ . This index is highly cited and popular among political parties wishing to correct the deficiencies of their country's electoral system as it carries the moral appeal of inequity reduction among parties within parliament (but ignoring outsiders). Considering a variety of moral criteria for PR implementation, [Taagepera and Grofman \(2003\)](#) collect two dozens inequity indices and compare their properties to select a handful of best candidates, guided by intuition. Likewise, Gallagher feels *Sainte Laguë*'s to be "soundest at theoretical level".

Our constitutionalist approach directly arrives at the same conclusion: as recalled in the introduction, *constitutional political equality* puts the focus on citizens rather than MPs or parties. Political participation is then uniquely measured by the electoral worth  $w$ , the fraction of representatives a citizen contributes to her national parliament; its statistical dispersion around the ideal national mean  $\omega^*$  is naturally gauged by the variance  $\sigma^2$ . As shown above, the *Sainte Laguë* rule minimizes inequity. To allow for comparisons, we employ the scale free version, the *coefficient of variation*  $\lambda = \frac{\sigma}{\omega^*}$  which carries the following meaning: there is a 95% probability that a randomly chosen voter has an effective worth falling within  $\pm 2\lambda\%$  of the national mean  $\omega^*$ . Index  $\lambda$  is a standard economic inequality measure (cf. [Trapeznikova \(2019\)](#)) satisfying *anonymity* (party names do not matter), *population independence* (nor census population), *consistency* (a party splitting in two halves with the same worth for both does not matter) and the *transfer principle* fitted to the PR problem: swapping a seat from a favored party to a disfavored one reduces

<sup>19</sup>[Rae \(1967\)](#) previously uses  $\frac{1}{n}$  instead of  $\frac{1}{2}$  in  $\Omega$ , an index introduced in sociology by [Jahn et al. \(1947\)](#) to measure segregation; the latter is in turn a rescaling of [Pietra \(1915\)](#)'s mean average dispersion (MAD) of income.

inequity (as long as advantage is not flipped as recalled by [Goldenberg and Fisher \(2019\)](#)).<sup>20</sup> Lastly, after some algebraic manipulations, we derive an original *decomposability* property, similar to the HHI industry concentration index used by antitrust authorities (cf. [Pindyck \(2017\)](#)).

**Result 3** *The squared inequity index sums the contributions to inequity from each party (including those without representation), weighted by their share of the vote:*

$$\lambda^2 = \sum_i \frac{v_i}{v_0} \left( \frac{\omega_i}{\omega^*} - 1 \right)^2 \quad (1)$$

We may thus pinpoint the exact sources of inequity among parties and their voters. An often major component of  $\lambda^2$  is the zero worth of voters endorsing a fringe party truly representing their political views. Appendix [A.1](#) details the apportionment inequity of the European Parliament seat distribution among the member states of the European Union while Appendix [A.2](#) provides a step-by-step construction of  $\lambda$  for the latest general election in Luxembourg.

The only formal drawback of the inequity index  $\lambda$  is the absence of an upper bound; note however that the 100% threshold, which is never surpassed in our large sample of elections and apportionments, corresponds to the highly recognizable situation where a pairwise duel is won by a razor-thin margin (e.g., a “swing” US congress seat). Indeed, in a single seat constituency under FPTP, the winner has  $v_1$  votes so that

$$\lambda^2 = \frac{v_1}{v_0} \left( \frac{v_0}{v_1} - 1 \right)^2 + \frac{v_0 - v_1}{v_0} (0 - 1)^2 = \frac{v_0}{v_1} - 1 \quad (2)$$

which may diverge if  $v_1$  is small.<sup>21</sup> In the most extreme situation where the winner had just a one more vote,  $v_1 = \frac{1}{2}v_0 + 1$  and  $\lambda^2 \approx 1$ . If such “swing” duels were ubiquitous, national inequity might achieve the 100% mark. In actual elections though, each side has pockets of regional dominance so that inequity falls below 100%, even when a so-called “landslide” takes place.

Our last theoretical contribution stems from the following consideration: assume each MP is elected in a single-seat district by a unanimous vote except for a share  $\hat{x} = \frac{\lambda^2}{1+\lambda^2}$  of the electorate that has been disenfranchised (barred from participating). The inequity will be exactly  $\lambda$  as may be checked from equation (2). We bring together the properties obtained so far:

**Result 4** *Electoral inequity is best gauged by the worth’s coefficient of variation  $\lambda$  satisfying*

- *$\lambda$  can be computed for any election.*
- *$\lambda$  can be computed for the apportionment of representatives among constituencies.*

<sup>20</sup>Be mindful that in  $\lambda$  or  $\Psi$ , we may neither transfer  $p$  nor  $w$ , unlike \$ in [Dalton \(1920\)](#)’s income distribution problem. Also, unlike in economic settings,  $\lambda$  is sensitive to house size. Indeed, for any PR method, a twice bigger parliament does not return twice as many seats to all parties, hence  $\lambda$  will be changed.

<sup>21</sup>A practical upper bound is the extremely unlikely situation where the winner beats 4 contenders with solely 20% of the vote,  $\lambda = \sqrt{\frac{4/5}{1/5}} = 200\%$ . This also provides a practical maximum for Gallagher’s  $\rho = \sqrt{\frac{(1-.2)^2 + 4 \times 0.2^2}{2}} \approx 63\%$ .

- $\lambda$  serves to compare elections across time and countries.
- $\lambda^2$  decomposes into a sum of contributions from parties or districts.
- $\frac{\lambda^2}{1+\lambda^2}$  represents the share of “disenfranchised” voters in the election.

## 2.6 Alternative notions of inequity

Our definition of electoral inequity has implicitly endorsed worth as the central object and its variance as the sole acceptable global measure of its dispersion in a population. As recalled by [Wada \(2012\)](#) for the well regarded divisor methods, plenty of alternatives exists within this large family and each achieves a reasonable objective. Our have rooted for worth as the constitutionally mandated object of interest when it comes to discuss electoral equity (or its absence). On a more technical level, variance is not the only statistical instrument to assess dispersion. A popular alternative is [Gini \(1912\)](#)’s coefficient  $\Gamma$  of income inequality, equal to the expected income difference of two randomly drawn citizens. [Van Puyenbroeck \(2008\)](#) applies this concept to electoral worth (where additional constraints must be accounted for) and finds  $\Gamma \propto \sum_{i,j} v_i v_j |w_i - w_j|$  which, luckily, is minimized by *Sainte Laguë*.<sup>22</sup> The Gini coefficient is thus a fully consistent alternative to  $\lambda$  for gauging electoral inequity as it also satisfies most of the previously seen properties. Empirically, the correlation coefficient between  $\lambda$  and  $\Gamma$  over some 500 elections reaches 95% which means that our inequity rankings would be extremely close should we favor  $\Gamma$  over  $\lambda$  (the linear fit is  $\Gamma \approx \frac{1}{2}\lambda$ ); beware though that two elections may still be ranked differently by  $\Gamma$  and  $\lambda$ . At the outset, we favor  $\lambda$  because  $\Gamma$  fails *decomposability* which is fundamental to uncover the inequity roots of a particular election.<sup>23</sup>

The Gini coefficient for the variable “party effective quotient” is  $\sum_{i,j} a_i a_j |p_i - p_j|$  as shown by [Van Puyenbroeck \(2008\)](#); unsurprisingly, it is minimized by the *American* PR rule since swapping seats among parties would increase the objective. Again, because a Gini coefficient is not decomposable among parties, the normalized variance  $\frac{\Psi}{p^*}$  would be a better decomposable measure of inequity between members of parliament (if this was chosen to be the ultimate gauge of constitutional equality).

A last alternative measure of dispersion, though not as popular as variance or Gini, satisfying the same desirable properties is [Theil \(1967\)](#)’s entropy  $\tau \propto \sum_i v_i w_i \ln w_i = \sum_i a_i \ln(a_i / v_i)$ . [Kamahara et al. \(2021\)](#) use this statistical measure of closeness between two distribution to decompose malapportionment along federal and provincial levels. There is however no reason for us to employ  $\tau$  as it is unrelated to electoral worth or any other ethical measure of interest.

<sup>22</sup>Indeed, the auction equilibrium condition states that whatever pair  $(i, j)$  is chosen, the difference  $w_i - w_j$  cannot be reduced by swapping a seat between these two parties.

<sup>23</sup>Beware that decomposability here differs entirely from the term used in income inequality studies. For instance the income Gini for the EU is 39% while the average of the within county Gini is 31% and the Gini among countries is 21%, hence the *intra* and *inter* figures do not sum to the global one.

Lastly, it is worth repeating that the 3 classic alternatives to *Sainte Laguë* also minimize a variance, differing only with respect to the variable of interest as we saw in Result 2.

### 3 Empirical Results

We first recall a number of court ruling highlighting the preeminence of the citizen/voter for the assessment of political equality. We then compute apportionment inequity for an exhaustive set of 40 competitive democracies, finding large variations. Both apportionment and the seat allocation method matter to test the equity of an electoral system. Our final effort brings under the same hood the majoritarian and PR systems so as to compare some 500 elections across 80 competitive democracies over 3 decades.

#### 3.1 Antecedents

As recalled in the introduction, the “political equality” awarded to adults by constitutions should go beyond the casting of a vote and instead make sure every vote weights equally upon the election result. Otherwise, people may cease to participate, endorse a populist view aiming to radically transform “the system” or vote untruthfully (strategically) for a second-best candidate (cf. [Duverger \(1951\)](#)’s psychological factor), none of which seems desirable. Because of the pervasiveness of this perceived debasement injustice, constitutional courts have been called to the rescue.

Regarding federal *apportionment*, Article 1.2 of the US constitution stipulates that “representatives shall be apportioned among states according to their respective numbers”; all four rules studied before are acceptable for that task and the US supreme court has abstained from endorsing any on grounds that none is perfect and that the constitution remains too vague. Elsewhere, constitutional courts often maintain the status-quo when asked for redress by aggrieved citizens with the following argument: any redress of the claimants’ low worth will necessarily reduce other citizens’ worth, a move which by itself would violate the court’s duty to uphold citizen equality.

However, when called to judge state *apportionment*, the US supreme court [SCOTUS \(1964\)](#) made the landmark ruling that “as nearly as is practicable, one man’s vote in a congressional election is to be *worth* as much as another’s” (emphasis added). Presented with single-seat districts of widely varying populations, the court dug into the 1787 convention to argue for constitutional protection of citizens against debasement and enforced its opinion with compulsory district re-drawing, mandating near identical populations across the entire United States.<sup>24</sup>

Concomitantly, the [UN \(1966\)](#) adopted a convention pledging “the right to vote and to be elected at genuine periodic elections which shall be by universal and equal suffrage and shall be held by secret ballot, guaranteeing the free expression of the *will* of the electors” (art. 25, emphasis

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<sup>24</sup>The quotient range ( $\max_i p_i - \min_j p_j$ ) is used to judge the appropriateness of districting proposals. As shown by [Edelman \(2015\)](#), it does not always yield a unique minimizing allocation, an impediment to its practical use.

added). The renewal of the [Swiss \(1998\)](#) constitution made good of this pledge by formally introducing the *genuine* vote.<sup>25</sup> The Swiss constitutional court [BGE \(2003\)](#) was then quick to void some electoral codes for denying a genuine vote to supporters of small parties in small districts.<sup>26</sup> Likewise, the German basic law sets two electoral principles sequentially, firstly *electoral equality* of voters and then *equal opportunities* of parties (art. 3.1 & 21.1 in [Deutscher Bundestag \(1949\)](#)). On that basis, the German constitutional court [FCC \(2008\)](#) stroke down the electoral law and mandated government to make sure that voters would have the “same legal chance of success” i.e., influence on the result of the federal legislative election.

We believe these judicial developments elevate the (voting) citizen above the elected representative (or party); furthermore, they judge an electoral code by its ability to endow citizens with an *effective, genuine* or *worthy* vote. We now assess how far actual systems go towards that goal.

### 3.2 Apportionment

Whereas apportionment is a long standing feature of US politics at the federal level, it has remained a relatively minor theme in the rest of the world; its impact over general electoral inequity has therefore been underrated. The fact that a Soria resident in Spanish Castile weights as much as 4 Madrid city dwellers is shocking to urbanites but as depopulation shrinks Soria’s influence, there is much political pressure to maintain the regressive apportionment rule rather than improve it. Similar discrepancies exists in all major democracies (e.g., Japan or Canada) to guarantee a minimal representation to sparsely populated territories.

Table 1 reports the inequity relative to population apportionment between districts (either single or multi-seats) in 40 countries (identified by their ISO3 codes) for the latest officially published allocation of seats (and the closest population census figures). A perfect zero score obtains when the entire country is established as a single super-district; this “at-large” solution was chosen by the [Netherlands \(1917\)](#), [Israel \(1920, under British mandate\)](#), [Uruguay \(1925\)](#) or [Serbia \(1990\)](#). Otherwise, the most equitable countries are Sweden and Poland due to a large house size (high average worth) and the continuous adjustment of district boundaries. The good standing of the USA is a direct consequence of the aforementioned supreme court decision that forced (nearly) equal population single seat districts.

At the other extreme (last column of in Table 1), some countries display a large apportionment inequity indicating a severe divergence from demographic proportionality. In some cases, there is a purposeful award of representatives to either minorities, lowly populated wilderness regions (Tierra del Fuego, Amazon, Khasmir), islands or foreign enclaves while some countries maintain

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<sup>25</sup>In article 34(2), the German original “unverfälschte” means unadulterated but is officially translated as “genuine” in English and “faithful” in French and Italian.

<sup>26</sup>The solution adopted was *biproportionality*, a modernized version a Dutch procedure used to fill seats in simultaneous proportion of party votes and regional populations as detailed by [Pukelsheim \(2014\)](#) and [Balinski \(2008\)](#).

ad-hoc of minima (Spain, Argentina) or even a maximum (Brazil). In last placed Colombia where an Amazon resident weights 20 Bogota city dwellers, the resulting high inequity has the following interpretation: out of 5 provinces, 4 are identically (over) represented and one is completely excluded. The majoritarian countries, to be discussed hereafter, are found in all columns of this table which means that apportionment is not a crucial feature of this electoral system.

Country	Worth	$\lambda$	Country	Worth	$\lambda$	Country	Worth	$\lambda$
POL 2011	12.2	1.9	AUS 2019	9.2	9.9	NIC 2011	11.5	25.2
SWE 2018	41.6	2.6	DNK 2015	23.9	11.6	TUN 2014	17.4	27.1
IRL 2016	33.7	3.0	GBR 2017	13.9	12.9	CHL 2014	8.7	31.4
BEL 2020	13.0	3.6	DEU 2015	3.6	13.8	JAP 2012	2.3	32.0
MEX 2015	2.7	3.6	LKA 2014	13.1	15.5	ESP 2019	7.5	32.5
ZAF 2011	3.9	4.3	RUS 2016	2.0	17.2	AFG 2018	25.3	38.4
IRQ 2011	9.6	4.5	TUR 2014	7.7	17.4	IND 2014	0.7	38.7
KOR 2018	5.0	5.0	IDN 2014	4.5	17.8	ARG 2010	6.9	41.6
POR 2019	22.0	5.8	COD 2018	12.4	18.1	LBN 2017	29.2	45.5
USA 2017	1.3	6.8	PER 2017	4.4	19.5	BRA 2010	2.8	45.6
NGA 2006	2.6	7.7	CAN 2015	9.9	21.7	PHL 2015	2.4	47.3
ROM 2011	15.3	7.8	ITA 2011	10.2	23.7	TWN 2019	4.0	53.1
CHE 2015	24.9	8.3	KEN 2009	6.1	24.0	MAR 2014	9.0	55.0
HUN 2018	13.4	9.0	FRA 2012	8.8	24.7	COL 2018	3.3	80.2

Table 1: Apportionment Inequity wrt. Population

The case against malapportionment remains potent as argued by [Dahl \(1983\)](#) and [Christiano \(2008\)](#) (among others); it resurfaces in the political sphere each time a *minority* of the people has the ability to block an important piece of legislation supported by a *majority* of the people. Reacting to this unsettling weakness, some countries have set-up an independent commission tasked with designing districts containing roughly the same population, aggregating or cutting through historical boundaries as needed prior to elections. Many US states have similar commissions but in that case to avoid the related phenomenon of *Gerrymandering*.

### 3.3 Majoritarian vs. Proportional Representation

Almost by design, majoritarianism is inequitable since it returns ample majorities to secure a stable government. Fictionally, Whigs, with as little as 26% of popular support against 74% for Tories, may win half of the parliament seats with a razor-thin margin and lose the other half by a landslide, if their supporters are well distributed in the districts to be won. This “inversion” phenomenon has been observed empirically for long and has even transmuted into the so-called *Gerrymandering* whereby the side controlling apportionment explicitly redraws constituencies to engineer a global

victory from a clear minority of the vote. When many parties concur to an election (with or without run-off), even lower relative majorities may win single-seat districts if the opposition fails to coalesce and remains divided. The phenomena runs deep: in the 2005 election, only one in five registered UK voters renewed support for the Blair government and only one in seven actively supported the French president's party in 2012; in both cases, the incumbent government won an absolute majority in parliament. Such stylized facts however fail to account for the entire spectrum of voters. Our electoral inequity measure is a remedy allowing comparisons across time and space.

We find inequity in the UK to be large with  $\lambda \approx 50\%$ , because liberals and the far-right get very few seats from their many votes as their support is scattered over the country with few pockets of dominance (as opposed to Labour and Tories). Inequity fell to  $\lambda = 35\%$  in the snap election of 2017 because of strategic voting for the 2 main parties. In France, the second round of legislative elections frequently pits just two candidates so that inequity is intermediate ( $\lambda \approx 33\%$ ) as these duels involve the major parties. The picture completely changes once we match the first round votes, reflecting the true preferences of French voters for a dozen parties, to the final composition of parliament. The staggering 1993 maximum  $\lambda = 85\%$  is due to the landslide victory of the conservative alliance that, on the one hand, gave a worth twice as much as the national average to 43% of voters and, on the other hand, a zero worth to a full 25% of voters endorsing the populists and ecologists who were barred from entering parliament (since nowhere did they end up most voted).

As recalled by [Boix \(2010\)](#), PR replaced majoritarianism a century ago in many countries at the behest of incumbents becoming electorally vulnerable by the very application of majority voting; this background drift, in turn, was generated by the accelerated population growth of cities and the progress of enfranchisement (towards socialist minded people).<sup>27</sup> Because PR constituencies are pegged to historical territories, they carry a highly variable number of seats per the aforementioned reasons. In the smaller ones, the rounding of fair quotas becomes highly contentious both regarding apportionment and the election proper: a rural area affected by urban flight fears losing another MP in the next re-apportionment while in a mid-sized district, a party polling intermediate figures is always on the brink of obtaining or losing a single seat. The preferences towards majoritarianism or proportionality then become correlated with a party's distribution of support across the country.

### 3.4 Overall Electoral Inequity

Whereas we previously computed an ex-ante inequity for the apportionment of representatives (MP) between districts on the basis of population, we now apply the same method ex-post once people have voted in the legislative election. Table 2 reports the average country-inequity for parliamentary elections held since 1990 in 80 democracies, which is an exhaustive list of coun-

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<sup>27</sup>Also, the need to reach religious compromise and the fear of revolution (cf. [Acemoglu and Robinson \(2000\)](#)).



tries over 4 million inhabitants with minimal political competition. Since our inequity measure is highly dependent on the share of unsuccessful voters, existing election databases (e.g., [Álvarez-Rivera \(2020\)](#), [Carr \(2020\)](#), [CLEA \(2019\)](#)) were of little use as they quite often fail to report the scores of marginal parties. We draw instead from Wikipedia, in several languages, whose source in turn is most often the numerous data handbooks edited by [Nohlen \(2020\)](#).

Country	$\lambda$ (%)	$\frac{\lambda^2}{\lambda^2+1}$ (%)	Country	$\lambda$ (%)	$\frac{\lambda^2}{\lambda^2+1}$ (%)
Uruguay	9.8	1.0	Germany	27.1	6.8
Netherlands	12.9	1.6	Spain	27.4	7.0
Denmark	14.3	2.0	Malawi	27.8	7.2
Sweden	17.6	3.0	Switzerland	27.9	7.2
South Africa	19.3	3.6	Belgium	28.5	7.5
Austria	19.9	3.8	Ghana	28.6	7.6
Brazil	20.4	4.0	Burkina Faso	29.5	8.0
United States	20.5	4.0	Venezuela	29.7	8.1
Finland	20.7	4.1	Niger	30.2	8.4
Israel	22.4	4.8	Colombia	31.3	8.9
Nigeria	22.7	4.9	Greece	31.7	9.2
Morocco	22.8	4.9	Iraq	31.8	9.2
Norway	23.1	5.1	Malaysia	32.1	9.3
Bolivia	23.7	5.3	Kenya	33.0	9.8
New Zealand	24.1	5.5	Thailand	33.7	10.2
Portugal	24.2	5.5	Poland	33.9	10.3
Sri Lanka	24.3	5.6	Mexico	35.2	11.0
El Salvador	24.9	5.8	Costa Rica	35.3	11.1
Ireland	25.5	6.1	Indonesia	35.4	11.1
Pakistan	26.5	6.6	Tanzania	35.6	11.2

Table 2: Voter Electoral Inequity (1990-2020), best half

The most equitable countries, Uruguay and the Netherlands, employ a single countrywide district<sup>28</sup> while the Scandinavian followers use compensation seats to specifically reduce inequity; whenever a country index  $\lambda$  is below 20%, less than 4% of the voters are “disenfranchised”. We may thus say that countries in the first column of Table 2 treat their electorate fairly, thanks to large or even maximal district sizes. Beware though of the electoral threshold; in the Israeli case, the 3% threshold pushes the average inequity to 22% due to the presence of many unsuccessful parties polling non negligible figures (below 3%). Inequity however fell to 9% in 2020 when voters were called for the third time in a year; they clearly voted strategically to (hopefully) expedite gov-

<sup>28</sup>In these countries, fringe parties enter parliament depending on the threshold of valid votes required for seat distribution (e.g.,  $0.67\% = \frac{1}{H}$  in the Netherlands or  $3.25\% \approx \frac{4}{H}$  in Israel).

ernment formation. The Spanish system, although previously seen wanting for apportionment, performs relatively well (second column next to Germany and Switzerland) which may indicate an ingrained tendency for strategic voting. Countries in the second column generate an intermediate level of electoral inequity that “disenfranchises” about 8% of their electorate. The third column gathers countries with a sustained amount of electoral inequity that “disenfranchises” nothing less than 14% of the electorate.

Country	$\lambda$ (%)	$\frac{\lambda^2}{\lambda^2+1}$ (%)	Country	$\lambda$ (%)	$\frac{\lambda^2}{\lambda^2+1}$ (%)
Zambia	35.8	11.4	Paraguay	42.0	15.0
Australia	36.1	11.5	Japan	42.7	15.4
Liberia	36.6	11.8	Slovakia	42.8	15.5
Romania	36.9	12.0	Lebanon	43.5	15.9
Italy	37.1	12.1	Turkey	45.0	16.8
Czechia	37.2	12.2	Hungary	45.3	17.0
Chile	37.3	12.2	United Kingdom	46.5	17.8
Nepal	37.4	12.2	Ecuador	47.5	18.4
Dominican Rep.	38.1	12.7	Ukraine	48.2	18.9
Croatia	38.3	12.8	Guatemala	48.3	18.9
Serbia	38.8	13.1	Russia	49.8	19.9
Bangladesh	38.9	13.2	Senegal	50.4	20.2
Uganda	39.9	13.8	Mozambique	50.9	20.6
Taiwan	40.1	13.8	India	52.0	21.3
Angola	40.2	13.9	Argentina	55.7	23.7
Peru	40.3	14.0	Philippines	56.4	24.1
Korea	40.7	14.2	Tunisia	59.5	26.1
Canada	40.8	14.3	Nicaragua	61.0	27.1
Guinea	40.9	14.3	Madagascar	62.1	27.8
Bulgaria	41.5	14.7	France	63.1	28.5

Table 3: Voter Electoral Inequity (1990-2020), worst half

We observe in Table 3 that majoritarian systems display the largest inequities with France, India, Russia and the UK among worst offenders. In these countries, parties with a geographical homogeneous support base obtain few seats, even though they deserve many more (e.g., Liberals in the UK, Rassemblement National in France). Shockingly for such an old democracy, France is the least able to give people a voice; the high 63% inequity score means that as much as 27% of the electorate is virtually excluded (disenfranchised) from the electoral process, election after election. Turkey presents an interesting twist on PR: by setting a 10% threshold, it excludes all parties but the largest three, thus keeping regionalists at bay; this ultimately brings the country into the majoritarian club. Russia has also used a 7% threshold to similar effect in the past (when

elections were still competitive). An instance of abnormally high inequity may be observed in the 2018 Mexican election where a small christian party negotiated dozens of eligible candidates with the leading left wing coalition, resulting in more than 10% of seats for less than 2.5% of the vote; otherwise, Mexico treats its voters quite equitably.

At the outset, we note how many of the most inequitable countries (fourth column) feature a populist government berating liberal democracy and its individual freedoms, pushing instead for an authoritarian leadership and direct democracy, leaving a divided opposition with a diminished representation. In Britain and France, strong populist movements are underrepresented and kept in the opposition; there is little doubt that should they form government, they would emulate the aforementioned ones into tightening their grip in power by augmenting inequity rather than reducing it. India is a case in point where for long the Indian National Congress took advantage of an inequitable electoral framework to retain power; now that the BJP ousted them, they are severely under-represented, castigated by the very system they did not want to reform, further losing influence and seats in the last 2019 election (cf. [Girvin \(2020\)](#)). A final observation regarding our large sample of legislative elections is that inequity appears to be slow rising by about one percentage point every decade (p-value of 0.12 for the linear trend econometric test).

### 3.5 Inequity vs. Disproportionality

We now compare our constitutionalist inequity measure  $\lambda$  to [Gallagher \(1991\)](#) 's popular *disproportionality* index  $\rho$ . Since both  $\lambda$  and  $\rho$  are MSE associated to not-so distinct PR rules, they ought arrive at fairly similar conclusions. This is indeed what we observe on [Figure 1](#) over our large election sample: in most situations, an election fair among parties is also fair among voters and vice-versa (78% correlation); there are nevertheless disagreements.

The 2012 Japanese election, for instance, features a high inequity typical of majoritarian systems but is classified as the one of the most disproportional (it lies far above the diagonal) because the dominant LDP got 61% of the seats from 34% of the vote while the opposition Democratic Party got 12% of the seats from 18% of the vote. The criteria underlying  $\rho$  overemphasizes the undue advantage of the former party and the unfair disadvantage of the latter one. Both contribute large amounts of inequity but unsuccessful votes were few that year so that  $\lambda$  was 58% which is large but not unusual for a majoritarian system. Conversely, Argentina constantly produces high inequity but intermediate disproportionality (it lies far below the diagonal), this because a string of local parties each get one or two seats out of few votes, so that  $\rho$  only registers a tiny gap wrt. their fair share; nevertheless, at country level, millions of votes are either given too much or too little effective worth for the determination of parliamentary composition; inequity is thus very large.

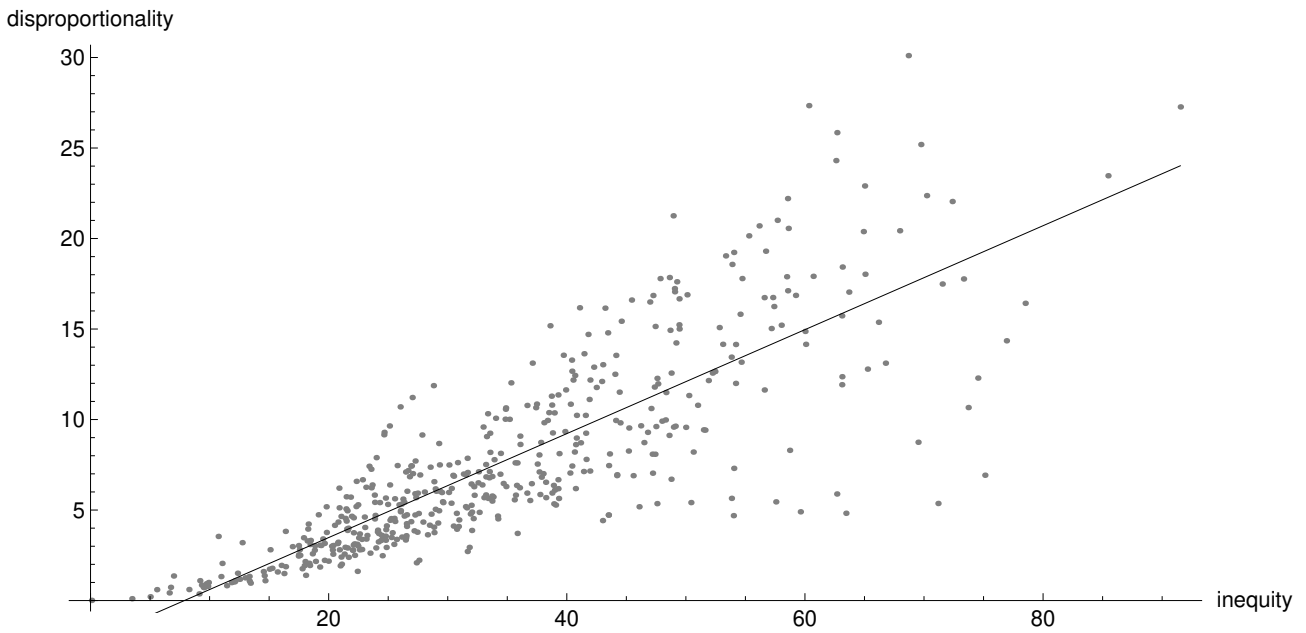


Figure 1: Comparing measures of Electoral Inequity

## 4 Conclusion

This article characterizes “genuine voter equality” in the competitive democracies adhering to the proportionality principle; we propose a natural measure of its inevitable dispersion which it then computed over a large sample of elections. For this, we employ a simple model encompassing the apportionment of districts and the election proper. We firstly characterize 4 classical vote-to-seat allocation rules as descending auctions, thus facilitating an intuitive understanding of their inner workings. Next, we derive the equity notion underlying each so as to judge their ability to bring about the alluded voter equality. We find Sainte Laguë to the best candidate, per the interpretation given by constitutional courts of the “electoral worth” notion. This specific rule then allows us to derive a natural index of inequity enjoying a clear meaning, and many axiomatic properties; it is applicable to all elections. We also trace the role of extreme apportionments (polar cases of majority and at-large elections) in producing or reducing global inequity.

In the empirical part, we first make explicit the inequitable representation of territories when seat allocation rules deviate from the constitutional mandate of population proportionality to advance geographical exceptions. We find this to be rather independent of whether the electoral system is majoritarian or PR. Next, the analysis of our large sample of elections confirms the suspicion that electoral inequity increases as we move along the PR axis from at-large towards majoritarianism. The value-added of our effort is to rank countries on that scale and provide an equivalent measure in terms of people virtually excluded from the franchise due to where they live and for whom they vote.

Regarding policy implications, if we aim to give all citizens the same voting chance, we can either grow the constituency size towards the super-district (e.g., from province to region) or change

the election rule from d'Hondt to Saint Laguë which treats fairly all parties (independently of their size). Such sweeping changes of the constitution however require a super majority that is difficult to put together since would-be losers anticipate their fate and will try to block the motion. A number of countries have adopted over recent decades a practical alternative known a *two-tier* system: add seats at the super-district level (e.g., Korea, Philippines) or at the regional level (e.g., Italy, Japan) and fill them with a second vote for national or regional parties (as opposed to individual candidates). Still simpler, Scandinavian countries allocate a limited number of (non territorial) *equalization seats*, typically as many as districts, to deserving parties in order to increase proportionality (without taking a second vote). A highly equitable alternative to these add-hoc fixes is *biproportionality* which is employed at regional level in Switzerland and the Netherlands.<sup>29</sup>

At the outset, we saw empirically that in too many countries a significant share of the electorate is “virtually” excluded from participation (i.e., their vote is ex-ante wasted); this situation may explain why voters who are increasingly taking notice turn towards populist parties claiming to “give people a voice” or take it to the streets to vent their anger (e.g., [French yellow vests](#)). Importantly, in “federal” countries aggregating heterogeneous cultures segregated across the geography, the majoritarian system still allows for political diversity with regionalist parties. The European parliament is a vivid example, proving that coalitions across countries and political views are feasible to advance ambitious goals. In the countries where a majority of the electorate (and an even larger share of the political establishment) nevertheless puts stability atop representativity, the electoral system need not slide towards “winner take all” majoritarianism; the constitution may be modified accordingly to make the rules of the political game more transparent to the public. An example is the German basic law’s frequent reference to “free democratic basic order” which the constitutional court balances again the ideal of electoral equality on a case-by-case basis.

When Churchill (1947) discussing a parliamentary act famously said that “democracy is the worst form of government except for all other forms”, he was advocating against encroachment of the majority since he had just uttered “it is not Parliament that should rule; it is the people who should rule through Parliament”.

## A Appendix

### A.1 European Parliament

The attributions of the European Parliament (EP) were initially so thin that the most populous and powerful countries did not mind awarding 6 MPs to tiny states, a PR breach bound to create a large inequity. The distortion has never been redressed as new members entered the union and parlia-

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<sup>29</sup>This mathematically more advanced method jointly respects the regional distribution of political support among parties AND the populations of districts; it increases the representation of deserving parties whose support is geographically scattered without deeply modifying the existing electoral framework.

ment acquired a key veto power over the design of new European legislation, although proposals have been made (cf. [Laslier \(2012\)](#)).

In 2019, the EU had 28 members while the EP hosted  $s_0 = 751$  seats/MPs representing over  $q_0 = 513$  million citizens i.e., an average worth  $w^* = 1.5$  seat per million people. Table 4 displays the worth  $w_i = \frac{s_i}{q_i}$  computed from population  $q_i$  and official seat allocation  $s_i$ . The “contr.” column shows the country inequity contribution  $\frac{v_i}{v_0} \left( \frac{\omega_i}{\omega^*} - 1 \right)^2$  whose sum is  $\lambda^2$ . While the Netherlands or Romania have a near ideal worth, Malta and Luxembourg enjoy too much and France and the UK are the most mistreated. Yet, summing individual worth across the large EU population reveals that the major deviations to the mean come from the over-represented. The resulting high 49% inequity may be pictured as follows: 80% of the European population is treated identically (and thus enjoy equal political rights) while the remaining 20% is disenfranchised (as if not worthy of citizenship). This is obviously an extreme interpretation of reality since the lowest worth is still 1.1 but it nevertheless reveals the extend of unfairness currently at work for the EP apportionment.

Unit	Worth	contr.	Unit	Worth	contr.	Unit	Worth	contr.
Malta	13.9	602.7	Bulgaria	2.4	54.6	Austria	2.1	27.0
Luxembourg	10.3	408.5	Croatia	2.6	50.0	Sweden	2.0	26.3
Lithuania	3.8	141.6	Hungary	2.2	41.1	Czechia	2.0	24.2
Cyprus	5.1	138.8	Slovakia	2.4	40.8	Greece	1.9	16.6
Latvia	4.1	121.5	Spain	1.2	40.4	Belgium	1.8	13.5
Estonia	4.6	114.0	Finland	2.4	38.4	Poland	1.3	6.7
Slovenia	3.8	105.5	Denmark	2.3	32.4	Romania	1.6	4.0
United Kingdom	1.1	82.9	Italy	1.2	32.4	Netherlands	1.5	0.4
Germany	1.2	69.6	Ireland	2.3	29.9			
France	1.1	66.5	Portugal	2.0	29.0		1.5	$\lambda = 48.6\%$

Table 4: Malapportionment in the European Parliament

## A.2 Luxembourg

We illustrate the construction of electoral inequity for the 2018 General Election in Luxembourg. Worth varies from 144 for Pirates to 343 for Christians. This raw measure is firstly relativized wrt. the country mean of 278. Taking the squared deviation, we see that one Pirate voter contributes 2335 units and one Christian voter 558. However, once we account for the number of voters of each party, Christians end up contributing 158 points of inequity against 151 points for Pirates. The global score is 486 which corresponds to  $\lambda = 22\%$ .

Party	Votes	%	Seats	Worth	R. worth	deviation	contribution	Fair seat
Christians	61,156	28.31	21	343	124	558	158	17.0
Socialists	38,020	17.60	10	263	95	28	5	10.6
Democrats	36,530	16.91	12	329	118	334	56	10.1
Greens	32,663	15.12	9	276	99	1	0	9.1
Nationals	17,887	8.28	4	224	81	380	31	5.0
Pirates	13,934	6.45	2	144	52	2,335	151	3.9
The Left	11,838	5.48	2	169	61	1,535	84	3.3
Total valid	216,024	100	60	278	100	-	<b>486</b>	$\lambda = 22\%$
Symbol	$v_i$	$\frac{v_i}{v_0}$	$s_i$	$w_i = \frac{s_i}{v_i}$	$\frac{\omega_i}{\omega^*}$	$(\frac{\omega_i}{\omega^*} - 1)^2$	$\frac{v_i}{v_0} (\frac{\omega_i}{\omega^*} - 1)^2$	$s_i^*$

Table 5: Luxembourg 2018 General Election

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