Duality of Welfare and Profit Maximization

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Abstract

Many economists are aware that the FOCs for efficiency and monopolization in a partial equilibrium framework are the extremes of the Ramsey (1927)-Boiteux (1956) FOC when the Lagrange multiplier for the budget varies. The object of this note is to formalize the duality between the welfarist and monopolist constrained maximization programs. We prove the following folk theorem:

$$\begin{array}{cc} \max & \text{Welfare} \\ \text{s.t.} & \text{profit} \geq \text{fixed cost} \end{array} \Leftrightarrow \begin{array}{cc} \max & \text{Profit} \\ \text{s.t.} & \text{output} \geq \text{minimum} \end{array}$$

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1 Introduction

The usual textbook characterization of efficiency in a partial equilibrium framework is to set price equal to marginal cost or, using the Lerner (1934) index of market power, to solve

$$\frac{p - C_m}{p} = 0 \tag{1}$$

Likewise, the characterization of monopoly pricing using the price elasticity of demand ε, is

$$\frac{p - C_m}{p} = \frac{1}{-\varepsilon} \tag{2}$$

In turn, the efficient Ramsey (1927)-Boiteux (1956) price for a regulated firm subject to a no-subsidy restriction is

$$\frac{p - C_m}{p} = \frac{1}{-\varepsilon} \frac{\lambda}{1 + \lambda} \tag{3}$$

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where λ is the Lagrange multiplier of the budget constraint imposed on the firm.

The similarity of these equations has escaped no one, especially the fact that equation (3) links (1) to (2) by sliding the multiplier from zero to infinity. Heuristically, if deficit is not an issue for the regulator then $\lambda = 0$ and we obtain marginal cost pricing while if deficit is an issue and is difficult to avoid (e.g., there are large fixed cost) then λ tends to infinity so that monopoly pricing becomes socially optimal.

Many economists are aware that the two maximization programs generating these equations seem to be dual one from another. The object of this note is to prove formally this folk theorem, absent from textbooks even the most advanced ones like Mas-Collel et al. (1995).

2 Formalization

We respectively denote W, W_D and W_S the welfare, consumer surplus and producer surplus. The decomposition $W = W_D + W_S = W_D + \Pi + F$ is at the heart of our method (fixed cost are left out of producer surplus). We start from a simple observation: a regulated monopoly sets prices to maximize profits Π under a restriction which can be expressed over consumer surplus ($W_D \geq \underline{W_D}$) or volume ($q \geq \underline{q}$) or price ($p \leq \underline{p}$). The social planner, on the other hand, sets prices to maximize welfare under a no-subsidy restriction.[1] The objective in the first program, profit Π, is the constraint of the second program. The duality would be obvious if the second program objective, welfare W, was the restriction of the first. This is not so but we are still able to show:

$$\begin{array}{l} \max W \\ \text{s.t. } \Pi \geq 0 \end{array} \Leftrightarrow \begin{array}{l} \max W \\ \text{s.t. } W_S \geq \underline{W_S} \end{array} \Leftrightarrow \begin{array}{l} \max W_S \\ \text{s.t. } W \geq \underline{W} \end{array} \Leftrightarrow \begin{array}{l} \max W_S \\ \text{s.t. } W_D \geq \underline{W_D} \end{array} \Leftrightarrow \begin{array}{l} \max \Pi \\ \text{s.t. } q \geq \underline{q} \end{array} \tag{4}$$

for judicious choices of the parameters. The first equivalence is trivial since the two programs only differ by the fixed cost constant. The last one is also obvious as consumer surplus is increasing in any of the market outputs. We thus only need to prove the middles ones.

Let $D_i(p_i)$ be the demand in sector $i \leq n$; it is assumed independent of prices in other sectors. The inverse function $P_i(q_i)$ is the willingness to pay (WTP) in the sector. Let $\mathbf{q} \equiv (q_i)_{i\leq n}$ denotes a bundle of quantities, $R(\mathbf{q}) \equiv \Sigma_{i\leq n} q_i P_i(q_i)$ the market revenue and $B(\mathbf{q}) \equiv \Sigma_{i\leq n} \int_0^{q_i} P_i(x)\,\mathrm{d}x$ the benefit or gross consumer surplus. The variable cost function is $C(\mathbf{q})$ (it depends only on total quantity if the goods are homogeneous). Welfare is $W(\mathbf{q}) = B(\mathbf{q}) - C(\mathbf{q}) = W_D(\mathbf{q}) + W_S(\mathbf{q})$ with $W_D(\mathbf{q}) = B(\mathbf{q}) - R(\mathbf{q})$ and $W_S(\mathbf{q}) = R(\mathbf{q}) - C(\mathbf{q})$.

Let $\Pi^M \equiv \max_{\mathbf{q}} R(\mathbf{q}) - C(\mathbf{q})$ and $\mathbf{q}^M$ be the argmax; let also $W^* \equiv \max_{\mathbf{q}} B(\mathbf{q}) - C(\mathbf{q})$ and $\mathbf{q}^*$ be

[1] Public funds are a source of inefficiency as shown originally by Vickrey (1955) with the concept of marginal cost of public funds.

the argmax. We consider two parametrized optimization programs. The welfarist program is

$$P^W(\alpha) \equiv \begin{cases} \max_{\mathbf{q}} B(\mathbf{q}) - C(\mathbf{q}) \\ \text{s.t. } R(\mathbf{q}) - C(\mathbf{q}) \geq \alpha \end{cases} \tag{5}$$

with solution $\mathbf{q}_\alpha$, value W_α and lagrange multiplier λ_α.

The capitalist program, which is the dual of the previous, is

$$P^M(\beta) \equiv \begin{cases} \max_{\mathbf{q}} R(\mathbf{q}) - C(\mathbf{q}) \\ \text{s.t. } B(\mathbf{q}) - C(\mathbf{q}) \geq \beta \end{cases} \tag{6}$$

with solution $\mathbf{q}_\beta$, value Π_β and lagrange multiplier λ_β. Observe that for $\underline{\alpha} \equiv R(\mathbf{q}^*) - C(\mathbf{q}^*)$, the solution of $P^W(\underline{\alpha})$ is $\mathbf{q}^*$ and $W_{\underline{\alpha}} = W^*$. Likewise, for $\bar{\beta} \equiv B(\mathbf{q}^M) - C(\mathbf{q}^M)$, the solution of $P^M(\bar{\beta})$ is $\mathbf{q}^M$ and $\Pi_{\bar{\beta}} = \Pi^M$. Both of these claims are true because $\mathbf{q}^*$ and $\mathbf{q}^M$ are unconstrained maximisers to which we add a constraint they satisfy.

The FOCs for the maximization of $P^W(\alpha)$ are $\forall i \leq n$,

$$B_i' - C_i' + \lambda_\alpha \left(R_i' - C_i'\right) = 0 \Leftrightarrow (1 + \lambda_\alpha)\left(P_i - C_i'\right) = -\lambda_\alpha q_i P_i' = \frac{\lambda_\alpha P_i}{\varepsilon_i} \tag{7}$$

where ε_i is the price elasticity of demand in sector i. Introducing further the Lerner index of market power $\mathcal{L}_i \equiv \frac{P_i - C_i'}{P_i}$, we obtain

$$\forall i \leq n, \mathcal{L}_i \varepsilon_i = \frac{\lambda_\alpha}{1 + \lambda_\alpha} \tag{8}$$

Likewise, the analysis of the FOCs of $P^M(\beta)$ yield

$$\forall i \leq n, R_i' - C_i' + \lambda_\beta \left(B_i' - C_i'\right) = 0 \quad \Leftrightarrow \quad \mathcal{L}_i \varepsilon_i = \frac{1}{1 + \lambda_\beta} \tag{9}$$

We can now state our claim:

Theorem 1 *Constrained welfare maximization is the dual of monopoly profit maximization under a consumer surplus restriction and vice versa.*

Proof We first show that $P^M(W_\alpha)$ has solution $\mathbf{q}_\alpha$ and value α. If the latter wasn't true, $P^M(W_\alpha)$ would achieve a value $\epsilon \neq \alpha$. Since $\mathbf{q}_\alpha$ satisfies $B(\mathbf{q}_\alpha) - C(\mathbf{q}^\alpha) = W_\alpha$ and $R(\mathbf{q}_\alpha) - C(\mathbf{q}^\alpha) \geq \alpha$, it must be true that $\epsilon > \alpha$. The solution of $P^M(W_\alpha)$, $\mathbf{q}^\epsilon$, therefore satisfies $B(\mathbf{q}^\epsilon) - C(\mathbf{q}^\epsilon) \geq W_\alpha$ and $R(\mathbf{q}^\epsilon) - C(\mathbf{q}^\epsilon) = \epsilon > \alpha$. Let us increase each q_i^ϵ by an equal amount Δq such that $\Delta R - \Delta C = \alpha - \epsilon < 0$ i.e., we construct $\tilde{\mathbf{q}} \equiv \mathbf{q}^\epsilon + \Delta q$ satisfying $R(\tilde{\mathbf{q}}) - C(\tilde{\mathbf{q}}) = \alpha$. As $\mathbf{q}^\epsilon$ solves $P^M(W_\alpha)$, it satisfies the FOCs (9) for some $\lambda_\beta > 0$, thus

$$\Delta R - \Delta C = \Delta q \left(\Sigma_{i \leq n} R_i'(q_i^\epsilon) - C_i'(q_i^\epsilon)\right) = -\lambda_\beta \Delta q \left(\Sigma_{i \leq n} B_i'(q_i^\epsilon) - C_i'(q_i^\epsilon)\right) = -\lambda_\beta (\Delta B - \Delta C)$$

which means that $\Delta B - \Delta C > 0$. We have thus constructed a bundle such that $B(\tilde{\mathbf{q}}) - C(\tilde{\mathbf{q}}) > W_\alpha$ and $R(\tilde{\mathbf{q}}) - C(\tilde{\mathbf{q}}) = \alpha$, contradicting the optimality of $\mathbf{q}_\alpha$ in $P^W(\alpha)$.

The proof that $P^W(\Pi_\beta)$ has solution $\mathbf{q}_\beta$ and value β is identical. If the latter wasn't true, the solution of $P^W(\Pi_\beta)$, $\mathbf{q}^\epsilon$ would satisfy $R(\mathbf{q}^\epsilon)-C(\mathbf{q}^\epsilon)\geq\Pi_\beta$ and $B(\mathbf{q}^\epsilon)-C(\mathbf{q}^\epsilon)=\epsilon>\beta$. Let us decrease each q_i^ϵ by an equal amount Δq such that $\Delta B-\Delta C=\beta-\epsilon<0$ i.e., we construct $\tilde{\mathbf{q}}\equiv\mathbf{q}^\epsilon-\Delta q$ satisfying $B(\tilde{\mathbf{q}})-C(\tilde{\mathbf{q}})=\beta$. As $\mathbf{q}^\epsilon$ solves P^W, it satisfies the FOCs (7) for some $\lambda_\alpha>0$, thus

$$\Delta B-\Delta C=\Delta q\left(\Sigma_{i\leq n}B_i'(q_i^\epsilon)-C_i'(q_i^\epsilon)\right)=-\lambda_\alpha\Delta q\left(\Sigma_{i\leq n}R_i'(q_i^\epsilon)-C_i'(q_i^\epsilon)\right)=-\lambda_\alpha\left(\Delta R-\Delta C\right)$$

which means that $\Delta R-\Delta C>0$. We have thus constructed a bundle such that $R(\tilde{\mathbf{q}})-C(\tilde{\mathbf{q}})>\Pi_\beta$ and $B(\tilde{\mathbf{q}})-C(\tilde{\mathbf{q}})=\beta$, contradicting the optimality of $\mathbf{q}_\beta$ in $P^M(\beta)$.

The second equivalence in (4) has therefore been established. To prove the third equivalence in (4), we observe that

$$\begin{array}{l}\max W_S\\ \text{s.t. } W_D\geq\underline{W_D}\end{array}\quad\Leftrightarrow\quad\hat{P}(\gamma)\equiv\begin{cases}\max_{\mathbf{q}}R(\mathbf{q})-C(\mathbf{q})\\ \text{s.t. } B(\mathbf{q})-R(\mathbf{q})\geq\gamma\end{cases}$$

with solution $\mathbf{q}^\gamma$ and value Π^γ. To end the proof, we will prove that $\hat{P}$ is dual of P^M and of P^W.

We need to show that $P^M(\Pi^\gamma+\gamma)$ has solution $\mathbf{q}^\gamma$ and value Π^γ while $P^W(\Pi^\gamma)$ has solution $\mathbf{q}^\gamma$ and value $\Pi^\gamma+\gamma$. We start with the latter and assume to the contrary that $P^W(\Pi^\gamma)$ has solution $\mathbf{q}^\epsilon$ and value $\epsilon>\Pi^\gamma+\gamma$. We use the continuity of all functions to make sure that the solution of $P^W(\Pi_\gamma)$ saturates the constraint i.e., $\mathbf{q}^\epsilon$ satisfies $R(\mathbf{q}^\epsilon)-C(\mathbf{q}^\epsilon)=\Pi_\gamma$. Since $B(\mathbf{q}^\epsilon)-C(\mathbf{q}^\epsilon)=\epsilon>\Pi^\gamma+\gamma$, we also have $B(\mathbf{q}^\epsilon)-R(\mathbf{q}^\epsilon)>\gamma$. Then, we subtract a uniform amount to create $\tilde{\mathbf{q}}$ such that $B(\tilde{\mathbf{q}})-R(\tilde{\mathbf{q}})=\gamma$. Now, the above reasoning based on the FOCs (7) implies that $\Delta R-\Delta C>0$, thus $R(\tilde{\mathbf{q}})-C(\tilde{\mathbf{q}})>\Pi_\gamma$, the desired contradiction. The proof for the other equality works likewise. ■

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