Relaxing Quality Differentiation through Capacity Limitation: A Note*

X. Y. Wauthy† and N. Boccard‡

March 2016

Abstract

We consider a duopoly stage-game where an incumbent sells a high quality product while enjoying an ample production capacity. We study the quality-capacity best response of an entrant, before price competition takes place. We partially characterize equilibrium prices and payoffs in the corresponding Bertrand-Edgeworth pricing games and show that the entrant tends to rely exclusively on capacity limitation in a subgame perfect equilibrium, thereby showing that vertical differentiation is not robust to Bertrand-Edgeworth competition.

Keywords: Capacity Constraint, Quality, Differentiation, Bertrand-Edgeworth Competition

JEL Classification: D43, L13, L51

---

*This paper is a revised and shortened version of a CORE DP 2009/50. We are grateful to seminar participants at ECARES, Ecole Polytechnique, UAB, University of Valencia and EARIE for comments. We thank Paul Belleflamme for comments on a preliminary draft and a referee of this journal for his constructive comments. We retain responsibility for any remaining errors.

†Université Saint-Louis Bruxelles, CEREC and CORE. Email : xavier.wauthy@usaintlouis.be.

‡Econ Dep, FCEE, Universitat de Girona, 17071 Girona, Spain. Financial support from Generalitat de Catalunya (AGAUR, XREPP) and Ministerio de Economía y Competitividad (contract ECO2013-45395-R) are gratefully acknowledged.

1 Introduction

Building on the original intuition of Edgeworth (1925) and the preliminary results of Levitan and Shubik (1972), the seminal paper of Kreps and Scheinkman (1983) analyses the role of capacity constraints under price competition with homogeneous goods. No comparable analysis has been performed for the case of differentiated goods. As discussed in Wauthy (2014), very little is known about the structure of Nash equilibria in Bertrand-Edgeworth differentiated industries. The aim of this note is to offer a theoretical contribution to this field of research and to show that the presence of capacity constraints may drastically impinge on firms incentives to differentiate their products.

To this end, we consider a market where an incumbent sells a high quality product and enjoys an arbitrarily large production capacity. We study the entry strategy of a challenger who may install a limited production capacity and commit to some degree of product differentiation. After entry has taken place, firms simultaneously set prices. We show that there exists a subgame perfect equilibrium of our stage-game in which the entrant chooses *not to differentiate by quality but relies exclusively on capacity commitment* to optimally relax price competition.

This note, though mainly of a technical nature, nevertheless offers a methodology to identify the structure of equilibrium prices and payoffs in price subgames with capacity constraints and differentiated products. Our results should prove useful in better understanding the nature of price competition in those, empirically relevant, markets where product differentiation and various forms of decreasing returns to scale co-exist.

2 Preliminaries

2.1 The model

We follow the Mussa and Rosen (1978) setup popularized by Tirole (1988) to model quality differentiation. A population of consumers with personal characteristic x is considered. The indirect utility function is given by $U(x, s, p) = xs - p$ when buying one unit of a product displaying quality s. Characteristics are uniformly distributed in the interval $[0, 1]$ and the mass of consumers is normalized to 1. We study the following stage game G:

- Stage 0: the incumbent, i, selects a quality and capacity both equal to 1.
- Stage 1: an entrant e selects his quality $s \leq 1$ at no cost and capacity $k \leq 1$.

- Stage 2: firms compete simultaneously in prices and produce at no cost up to capacity. Producing beyond capacity is not feasible.

We denote $G(s,k)$ the pricing game occurring at the last stage. Our solution concept for the game G is *Subgame Perfect Nash Equilibrium*. Subgames $G(1,k)$ correspond to the Bertrand-Edgeworth games first studied by Levitan and Shubik (1972) whereas subgames $G(s,1)$ are studied in Choi and Shin (1992). Obviously, the subgame $G(1,1)$ is the standard Bertrand, homogeneous good, pricing game.

2.2 Demand Functions

Consumers make their choice at the last stage by comparing the respective surpluses they derive when buying from the incumbent, the entrant or when refraining from consuming i.e., they compare $x - p_i$, $xs - p_e$ and 0. The typical partition of the population as a function of optimal choices is obtained as follows. First, one identifies the so-called indifferent consumer. By definition, this consumer enjoys the same surplus when buying the high or low quality product. Formally $\tilde{x}(p_i, p_e)$ solves $x - p_i = sx - p_e$. We obtain

$$\tilde{x}(\cdot) = \frac{p_i - p_e}{1 - s}$$

Then we identify the marginal consumer who, by definition, is indifferent between buying quality product $j = i, e$ and refraining from consuming. Formally, $\overline{x}_j$ solves $xs_j - p_j = 0$. We obtain

$$\overline{x}_j = \frac{p_j}{s_j}$$

In the presence of differentiation ($s < 1$), it is a straightforward exercise to show that notional demands are given by

$$D_i(p_i, p_e) = \begin{cases} 0 & \text{if} \quad p_e + 1 - s \leq p_i \\ 1 - \tilde{x}(\cdot) & \text{if} \quad \frac{p_e}{s} \leq p_i \leq p_e + (1 - s) \\ 1 - \overline{x}_i & \text{if} \quad p_i \leq \frac{p_e}{s} \end{cases} \tag{1}$$

$$D_e(p_i, p_e) = \begin{cases} 0 & \text{if} \quad p_e \geq p_i s \\ \tilde{x}(\cdot) - \overline{x}_e & \text{if} \quad p_i - 1 + s \leq p_e \leq p_i s \\ 1 - \overline{x}_e & \text{if} \quad p_e \leq p_i - 1 + s \end{cases} \tag{2}$$

Two remarks are in order. First, whenever $s = 1$ i.e., whenever products are homogeneous, the demand functions (1) and (2) degenerate to the usual, discontinuous Bertrand demand functions. For simplicity we assume that consumers are split equally among the two firms in case of a price tie. Second, in subgame $G(s, 1)$, these functions define not only notional but effective demands resulting from consumers' optimal choices given prices and products' qualities. By contrast, in the price subgames $G(s, k)$ with $k < 1$, notional demands may or may not define consumers' actual consumptions since the entrant's capacity constraint may prohibit him from serving all consumers. Should that happen, some rationed consumers may turn to the incumbent. Accordingly, we shall distinguish in the forthcoming analysis demands, as expressed by consumers, from sales, as realized by the firms given the capacity limitation.

2.3 Sales Functions in the presence of rationing

If the entrant has built a limited capacity ($k < 1$), there exist prices levels leading up to more demand than can be served by firm e i.e., prices such that $D_e(p_e, p_i) > k$. In such cases, some consumers will be rationed and possibly report their purchase on the incumbent. In order to characterize firms' sales in that situation, we assume that *efficient rationing* is at work. Under this rule, rationed consumers are those exhibiting the lowest willingness to pay for the good. The limited k units sold by the entrant are contested by potential buyers and end-up being acquired by the most eager.[1]

Two configurations must be distinguished depending on whether prices are such that, according to consumers' optimal choices the market is shared between the two firms or preempted by the entrant.

In the case of duopoly competition, i.e. for price constellations such that both firms enjoy a positive demand, we identify a price threshold, ρ_e, above which the entrant's capacity is binding:

$$D_e(p_e, p_i) = \frac{p_i - p_e}{1 - s} - \frac{p_e}{s} > k \Leftrightarrow p_e < (p_i - k(1 - s))s \equiv \rho_e \tag{3}$$

In the monopoly case, i.e. for prices such that $D_i(\cdot) = 0$, we identify the following capacity binding threshold:

$$D_e(p_e, p_i) = 1 - \frac{p_e}{s} > k \Leftrightarrow p_e < s(1 - k) \tag{4}$$

Using (3) and (4), the entrant is capacity constrained i.e., $S_e(p_e, p_i) = k$, whenever

$$p_e \leq \min \{\rho_e, s(1 - k)\} \tag{5}$$

[1] A particular way to rationalize this rationing rule amounts to assume that a secondary market opens where consumers may take advantage of the arbitrage possibilities at no cost.

Now, using the notional demand (1), we obtain the residual demand addressed to the incumbent firm when the entrant rations as

$$D_i^r(p_i) \equiv 1 - ks - p_i. \tag{6}$$

Combining these expressions with consumers' demand defined by equations (1) and (2), we may express the firms' effective sales functions:[2]

$$S_e(p_i,p_e) = \begin{cases} 0 & \text{if } \; p_e \geq p_i s & (a) \\ \frac{p_i - p_e}{1-s} - \frac{p_e}{s} & \text{if } \; p_e \in [\max\{p_i - (1-s), \rho_e\}\,; p_i s] & (b) \\ 1 - \frac{p_e}{s} & \text{if } \; p_e \in [s(1-k); p_i - (1-s)] & (c) \\ k & \text{if } \; p_e \leq \min\{\rho_e, s(1-k)\} & (d) \end{cases} \tag{7}$$

$$S_i(p_i,p_e) = \begin{cases} 0 & \text{if } \; p_i \geq p_e + 1 - s & (a) \\ 1 - ks - p_i & \text{if } \; p_i \in \left[\frac{p_e}{s} + k(1-s); p_e + 1 - s\right] & (b) \\ 1 - \frac{p_i - p_e}{1-s} & \text{if } \; p_i \in \left[\frac{p_e}{s}; \frac{p_e}{s} + k(1-s)\right] & (c) \\ 1 - p_i & \text{if } \; p_i \leq \frac{p_e}{s} & (d) \end{cases} \tag{8}$$

We end-up with a partition of the price space characterized by piecewise linear sales functions. Starting from an arbitrarily high entrant price p_e, and given some incumbent price p_i, the entrant's sales are first equal to zero, then whenever sales become positive as a result of price decreases, they correspond to a situation where both firms enjoy positive market shares (7:b). When price decreases further, either the entrant monopolizes the market (if p_i is large enough), or he hits the capacity constraint. The corresponding sales segments for the incumbent are given by equation (8): starting from high prices where sales are nil, the incumbent starts to enjoy sales originating in the set of rationed consumers (8:b). Then when decreasing her price, she relaxes the capacity constraint for the entrant, and for sufficiently low prices, she simply pulls him out of the market and sells along her monopoly demand. Obviously, whenever $k = 1$, the sales function degenerate into the demand functions defined by (1) and (2).

3 Equilibrium analysis

With these sales functions in hand, we are now ready to study the prices subgames $G(k, s)$. The analysis proceeds in three steps. First, we characterize firms' best responses in subgames

[2]Notice that branch (7:c) is void if $p_i < 1 - ks$.

$G(s,k)$. Second, we characterize firms' payoffs in the price equilibria of $G(s,k)$. Third, we establish an upper bound for the entrant's payoff over the whole set of price subgames $G(s,k)$ which, finally, enables us to characterize the set of subgame perfect equilibria of G.

3.1 Price best responses

In the absence of production cost, firms' profits in the pricing game are

$$\Pi_e(p_i,p_e) = p_e S_e(p_i,p_e) \quad \text{and} \quad \Pi_i(p_i,p_e) = p_i S_i(p_i,p_e) \tag{9}$$

The presence of the capacity constraint introduces a novel strategic consideration into our inquiry. Whenever $k<1$, the analysis of $G(k,s)$ must take account of the possibility that the entrant's capacity is strictly binding and that the incumbent recovers rationed consumers. In such a case, optimal strategy amounts for the entrant to sell his capacity at the highest price[3]. On the other hand, the incumbent maximizes profits by acting as a monopolist along the residual demand. In this case, he collects her minimax payoff.[4]

Incorporating this argument into the traditional analysis, we may informally discuss the shape of each firm's best reply.

Consider first the entrant and concentrate on the price constellations such that the demand addressed to firm e is positive. The best this firm can hope is to share the market as long as it does not hit his own capacity. For price constellations such that the capacity is binding (relatively low p_e, relatively high p_i), the best for the entrant is to sell his capacity at the highest possible price. The entrant's payoff is concave throughout the domain so that the corresponding best response is continuous and kinked. A typical configuration is depicted as a bold solid line on Figure 1.

The best reply of the incumbent in the region where the capacity constraint is binding is to name the minimax price $\bar{p}_i \equiv \frac{1-ks}{2}$. In the region where the capacity is not binding, the incumbent either shares the market with the entrant or sets the limit price that excludes the entrant from the marketContrary to the case of the entrant, the rationed consumers recovered by the incumbent in the capacity binding region break the concavity of his payoff function. Accordingly, for many values of p_e, there are two local maxima of the incumbent profit function to be considered. It is then a matter of computations to show that there exists a critical level of p_e below which the minimax strategy strictly dominates the standard one

[3] which is ρ_e as defined by equation (3)

[4] Formally, by setting $\bar{p}_i \equiv \frac{1-ks}{2}$ she collects $\overline{\pi}_i \equiv \frac{(1-ks)^2}{4}$.

(i.e. when an incumbent faces a very aggressive entrant, she prefers to retreat on her residual market in order to enjoy a quasi-monopoly).

A typical shape for the corresponding best reply is depicted as a bold dashed line in Figure 1.

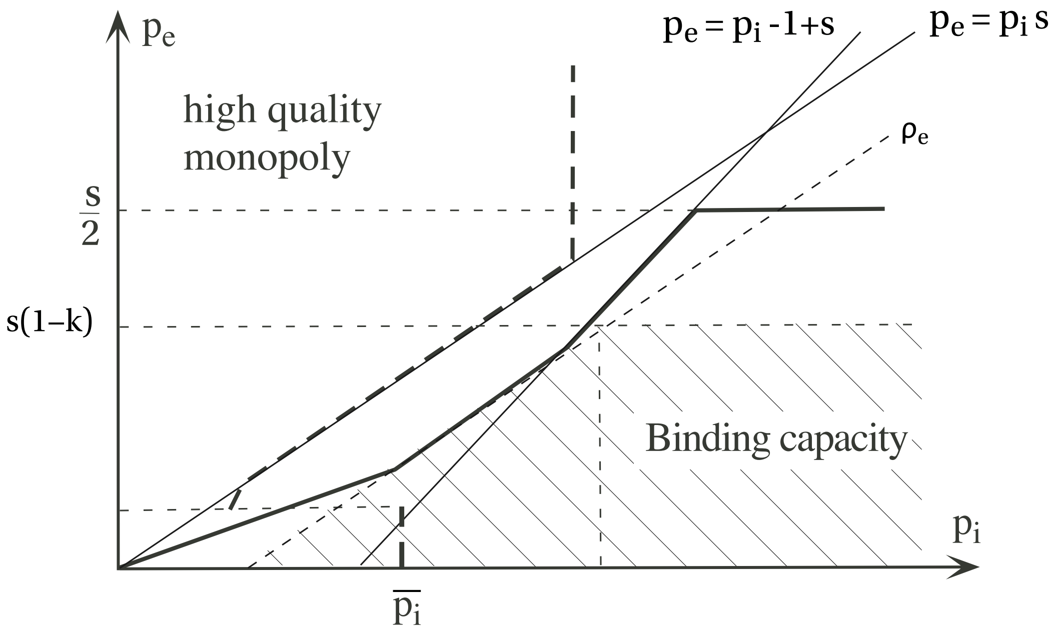

Figure 1: The price space with binding capacity

The complete formal derivation of the best replies is a bit tedious because different sub-cases must be considered depending on the relevant constellations of parameters (s, k). However, since we are ultimately interested in deriving optimal choices of capacity and quality levels, we have to cover all possible configurations. Moreover, it is worth mentioning that this characterization of best replies is, to the best of our knowledge, a new result which shall prove useful in any future analysis of Bertrand-Edgeworth pricing games under quality differentiation.

Regarding the entrant, along the segments (7:b,c), his best response is obtained from the first order conditions computed respectively along segment (7:b) with $\frac{p_i s}{2}$ or as a corner solution with $p_i - 1 + s$, and along the segment (7:c), the best response is the monopoly price $\frac{1}{2}s$. Plugging the constraints defining the domain of the definition of the various demand segments, we obtain:

$$BR_e(p_i,k) = \begin{cases} \frac{p_i s}{2} & \text{if} \quad p_i \le 2k(1-s) & (a) \\ \rho_e & \text{if} \quad 2k(1-s) \le p_i \le \min\left\{1-\frac{s}{2}, 1-ks\right\} & (b) \\ p_i - 1 + s & \text{if} \quad 1-ks \le p_i \le 1-\frac{s}{2} & (c) \\ \max\left\{\frac{s}{2}, s(1-k)\right\} & \text{if} \quad p_i \ge \min\left\{1-\frac{s}{2}, 1-ks\right\} & (d) \end{cases} \tag{10}$$

Notice that the entrant's payoff function remains concave in own prices (over the domain where $D_e(.) \geq 0$), hence the best response is a continuous function.

Regarding the incumbent, along segment (8:b) her best response is to name the minimax price $\bar{p}_i$. Along segment (8:c), the best response is given by the first order condition of the relevant payoff specification; computations yield $\frac{p_e+1-s}{2}$. Along segment (8:d), the monopoly price $\frac{1}{2}$ defines the best reply. As should appear from the inspection of $S_i(p_e, p_i)$, the payoff of the incumbent is likely to be non-concave when her sales switch from segment (8:b) to (8:c). Accordingly, the best response to p_e might be non-unique and we must formally compare the payoffs obtained under the minimax strategy with those prevailing under the traditional Bertrand competition. Solving $\pi_i\left(\frac{p_e+1-s}{2}, p_e\right) = \overline{\pi}_i$ for p_e, we obtain:

$$\hat{p}_e(s,k) \equiv \sqrt{1-s}\left(1-ks-\sqrt{1-s}\right) \tag{11}$$

Equation (11) defines the critical price level of the entrant below which the incumbent prefers to retreat on her protected residual market and name the minimax price, rather than competing upfront with the entrant by setting the market sharing price. We shall refer to this price as the *cut-off price*. Notice however that there may exist constellations where $\overline{\pi}_i > \pi_i\left(\frac{p_e+1-s}{2}, p_e\right)$ over the whole domain where $\frac{p_e+1-s}{2}$ is defined. In this case, the minimax strategy dominates the market sharing best reply candidate over the entire domain (8:c). If this is the case, we must compute the incumbent's payoff at the frontier of segment (8:c,d), i.e. for $p_i = \frac{p_e}{s}$. Solving $\frac{p_e}{s}\left(1-\frac{p_e}{s}\right) = \overline{\pi}_i$ for p_e, we obtain a second cutoff price :

$$\tilde{p}_e(s,k) \equiv \frac{s}{2}\left(1-\sqrt{ks(2-ks)}\right) \tag{12}$$

Lastly, we need to identify when each of these cut-off prices applies i.e., we solve $\hat{p}_e(\cdot) = \tilde{p}_e(\cdot)$ to obtain:

$$h(s) \equiv \frac{1}{s}\left(1-\frac{2\sqrt{1-s}}{2-s}\right) \tag{13}$$

Equation (13) partitions the capacity-quality space according to the definitions of the relevant cut-off price. Notice that this function is increasing in s, i.e. the larger s, the lower the degree of product differentiation and the larger the domain of capacities for which the first cutoff $\tilde{p}_e(s,k)$ applies.

Formally we have:

- if $k \geq h(s)$, then

$$BR_i(p_e) = \begin{cases} \frac{1-ks}{2} & \text{if} \quad p_e \leq \hat{p}_e & (a) \\ \frac{p_e+1-s}{2} & \text{if} \quad \hat{p}_e < p_e \leq \frac{1-s}{2-s}s & (b) \\ \frac{p_e}{s} & \text{if} \quad \frac{1-s}{2-s}s \leq p_e \leq \frac{s}{2} & (c) \\ \frac{1}{2} & \text{if} \quad p_e \geq \frac{s}{2} & (d) \end{cases} \tag{14}$$

- if $k \leq h(s)$, then

$$BR_i(p_e) = \begin{cases} \frac{1-ks}{2} & \text{if} \quad p_e \leq \tilde{p}_e & (a) \\ \frac{p_e}{s} & \text{if} \quad \tilde{p}_e < p_e \leq \frac{s}{2} & (b) \\ \frac{1}{2} & \text{if} \quad p_e \geq \frac{s}{2} & (c) \end{cases} \tag{15}$$

The critical values $\hat{p}_e$ and $\tilde{p}_e$ therefore identify the price levels at which the incumbent is indifferent between naming the security price $\bar{p}_i = \frac{1-ks}{2}$ or naming a lower price which ensures a larger market share. The resulting discontinuity is likely to destroy the existence of a pure strategy equilibrium.[5]

3.2 Price Equilibrium

We analyze the Nash equilibria for each price subgame $G(s,k)$. We shall consider in turn the two limit cases $G(1,k)$ and $G(s,1)$ before considering the interior cases where $k < 1$ and $s < 1$.

3.2.1 Equilibrium in $G(1,k)$ and $G(s,1)$

In $G(1,k)$, the vertical differentiation model degenerates into Bertrand-Edgeworth competition for a homogeneous product. Levitan and Shubik (1972) analyze this game under efficient rationing and derive the following result.[6]

[5]In order to avoid any misunderstanding, let us stress that it is only the existence of a *pure* strategy equilibrium which is problematic here. Since payoffs are continuous as long as products are differentiated, the existence of a mixed strategy equilibrium is ensured by Glicksberg (1952)'s theorem.

[6]Note that if we apply Gelman and Salop (1983)'s Stackelberg sequentiality to the current demand, we obtain exactly the same optimal capacity.

Lemma 1 *The game $G(1, k)$ has a unique price equilibrium in which the entrant earns exactly $k\tilde{p}_e(1, k)$. Furthermore the maximum of this payoff is $\pi_e^\dagger \equiv \frac{3}{4} - \frac{1}{\sqrt{2}} \simeq 0.043$ and is reached for $k^\dagger \equiv 1 - \frac{1}{\sqrt{2}} \simeq 0.293$.*

In $G(s, 1)$, the capacity constraint ceases to bind and we end-up in a standard Bertrand game under vertical differentiation. Choi and Shin (1992) analyzes this game and show the following result.[7]

Lemma 2 *For $s < 1$, the game $G(s, 1)$ has a unique pure strategy equilibrium:*

$$p_i^* = \frac{2(1-s)}{4-s} \text{ and } p_e^* = \frac{s(1-s)}{4-s} \tag{16}$$

The optimal quality for the entrant in the class of pricing games $\{G(s, 1), s < 1\}$ is $s^* = \frac{4}{7}$, yielding the profit $\pi_e^* = \frac{1}{48} \simeq 0.021$.

3.2.2 Price Equilibrium in $G(s, k)$

When products are differentiated and one firm faces a capacity constraint, three relevant equilibrium configurations exist. First, for a high enough capacity level the pure strategy Bertrand equilibrium is preserved as documented for instance by Canoy (1996). Second, for intermediate capacity levels, there exists a semi-mixed equilibrium where the entrant plays a pure strategy while the incumbent mixes over two atoms as documented by Krishna (1989). Third, for smaller capacity levels, fully mixed strategy equilibria i.e., equilibrium in which the two firms use non-degenerate mixed strategies, exist. To the best of our knowledge, no general characterization of an equilibrium exists yet for this last category of price subgames. It is a matter of algebra to identify the partition the capacity-quality space where each type of equilibria prevails.

Computations have been relegated to the appendix but the main intuitions can be summarized as follows. The presence of a capacity constraint allows the incumbent to contemplate a new strategy, namely setting a high price which creates rationing at the entrant's shop and therefore generates demand spillovers. The profitability of such a deviation to the minimax price depends on the size of the residual market. When the entrant's capacity is large enough, the residual market is too small and the incumbent is better off playing the standard Bertrand strategy i.e., "fight". Moreover, given the entrant's capacity, the standard equilibrium is more likely to be preserved if product differentiation is strong. Putting these

[7] Since these results are not knew, their proof has been omitted for the sake of brevity.

two forces together, we may identify a critical level of capacity (whose value depends on the degree of product differentiation) above which the presence of the capacity constraint does not affect equilibrium behavior. Notice that this bound $g(s)$ is strictly larger than the sales level of the incumbent at the equilibrium defined by Lemma 4 below.[8]

Whenever $k < g(s)$, a pure strategy equilibrium fails to exist. Two results can nevertheless be established. For intermediate capacities, there exists an equilibrium in which the incumbent randomizes over two atoms while the entrant plays the pure strategy $\hat{p}_e$.[9] However, there also exists a domain of small capacities where this equilibrium fails to exist. When this is the case, both firms use a non-degenerate mixed strategy in equilibrium. In any mixed equilibrium strategy, the following result holds:

Lemma 3 *Let $k < g(s)$. In equilibrium of $G(k, s)$, $p_e^- \leq \hat{p}_e$ if $k \geq h(s)$ and $p_e^- \leq \tilde{p}_e$ if $k \leq h(s)$. The entrant's equilibrium payoff is bounded from above by $k\hat{p}_e(s, k)$ if $k \geq h(s)$ and by $k\tilde{p}_e(s, k)$ if $k \leq h(s)$.*

The proof of this Lemma is developed in the Appendix.

3.3 Optimal Selection of Capacity and Quality

Although we do not have a characterization of the mixed strategy equilibrium for all possible subgames, we have derived enough information at this step to establish our main proposition. This proposition formalizes the existence of an optimal strategy that consists for the entrant to match the incumbent's quality and rely exclusively on capacity limitation to relax competition. In other words, this proposition states that as a mean to relax price competition in the last stage of the game, capacity limitation may dominate quality differentiation.

Proposition 1 *There exists an optimal quality-capacity pair $s = 1$ and $k = k^\dagger$.*

Proof For $k < h(s)$, we have that $\pi_e(F_e, F_i) \leq k\tilde{p}_e(s, k) = \frac{ks}{2}\left(1 - \sqrt{ks(2 - ks)}\right)$ which is a function of the product $x = ks$. Its maximum is reached for $x = k^\dagger$ and yields an overall

[8] Hence, this result can be viewed as the application for a case of vertical differentiation of the analysis developed in Benassy (1989) or Canoy (1996) for various cases of horizontal differentiation: given some degree of product differentiation, there exists a lower bound for the entrant's capacity above which the residual market is so small that it is not profitable for the incumbent to deviate from the Bertrand equilibrium price to the minimax strategy.

[9] We do not construct this equilibrium explicitly in the present paper. The interested reader is referred to Krishna (1989) for an early characterization. Boccard and Wauthy (2010a) offer a comparable characterization for a vertical differentiation set-up.

maximum profit $\pi_e^\dagger$. It then remains to observe that this is precisely the optimal quality and the maximum entrant's payoff for $s = 1$ and $k = k^\dagger$ as shown in Lemma 1. The maximum payoff over the domain $s < 1$ and $k < h(s)$ is therefore dominated by that in $G(1, k^\dagger)$.

A similar analysis applies for $s < 1$ and $h(s) \le k \le g(s)$. The upper bound, computed in the previous lemma, $k\hat{p}_e(s,k) = k\sqrt{1-s}\left(1 - ks - \sqrt{1-s}\right)$ reaches its maximum for $k = \frac{1-\sqrt{1-s}}{2s}$. Replacing by the optimal value and simplifying, the objective is now $\frac{\sqrt{1-s}\left(1-\sqrt{1-s}\right)^2}{4s}$. The maximum is achieved at $\bar{s} \equiv 2(\sqrt{2}-1) \simeq 0.83$ and leads to the optimal capacity $k^\dagger/\bar{s} \simeq 0.35$ and profit $\pi_e^\dagger$ exactly. We have thus shown that the entrant's profit for $h(s) \le k \le g(s)$ is lower than a function whose maximum is $\pi_e^\dagger$.

Finally, for $s < 1$ and $k \ge g(s)$, the optimum strategy is to differentiate with $s^* = \frac{4}{7}$ to earn $\pi_e^* = \frac{1}{48} \simeq 0.021 < \pi_e^\dagger \simeq 0.043$. Overall, the pair $\left(1, k^\dagger\right)$ is an optimal strategy. ■

Notice that other optimal quality-capacity pairs may exist; they necessarily satisfy $s \ge \bar{s}$ and $sk = k^\dagger$.

4 Comments

Proposition 1 has proven that quality imitation and an exclusive reliance on capacity constraint is an optimal course of action for the entrant. This finding therefore demonstrates that vertical differentiation is not robust to the presence of Bertrand-Edgeworth competition, at least as a means to relax price competition.[10] This result needs however to be qualified because it has been established in a highly stylized model. In particular, the efficient rationing rule and the fact that quality is not costly are instrumental in obtaining such a clearcut result. When quality costs are quadratic and sunk, it is easy to show that we do not end-up with a no-differentiation result. Still, the presence of capacity constraints clearly weakens the incentives to differentiate by quality.[11]

More generally, our analysis suggests that the presence of capacity constraints carries dramatic implications in models where product differentiation is endogenous. Within the limited scope of our model, the supposedly ubiquitous "principle of differentiation" ceases to hold. Whether such a conclusion carries on more generally is an open question which calls first for deeper investigation of the nature of equilibria in pricing games with differentiated products and capacity constraints.

[10] Boccard and Wauthy (2010b) establish a comparable result under Bertrand and soft capacity constraints as opposed to the present case of hard constraints known as Bertrand-Edgeworth competition.

[11] Actually a comparable tendency is observed if we allow for quality leap-frogging.

Appendix

A Proof of Lemma 4

We identify the quality-capacity constellations where a pure strategy equilibrium exists i.e., where the pure strategy equilibrium prevailing in the limiting case where $k = 1$ and identified in Lemma 2 is preserved.

Lemma 4 *For $s < 1$, the pair (p_i^*, p_e^*) is a pure strategy equilibrium of $G(s,k)$ whenever $k \geq g(s) \equiv 1 - \frac{4\sqrt{1-s}}{4-s}$.*

Proof Recall by Lemma 2 that p_i^* is a best response to p_e^* in the absence of capacity constraint. Hence, p_i^* remains a best response to p_e^* only if $p_e^* \geq \hat{p}_e(\cdot)$; a straightforward manipulation of this condition shows it is equivalent to $k \geq 1 - \frac{4\sqrt{1-s}}{4-s} = g(s)$.

Notice now that the following property holds: $g(s) > h(s) = \frac{1}{s}\left(1 - \frac{2\sqrt{1-s}}{2-s}\right)$ since this is equivalent to $16s^2(1-s) + s^4(3+s) > 0$ which is always true over the relevant domain $0 \leq s \leq 1$. We may now check that $\hat{p}_e(\cdot)$ was indeed the benchmark to use i.e., best reply (14) applies, not (15). ∎

B Proof of Lemma 3

The equilibrium strategy used by firm $j = i, e$ in equilibrium of $G(k,s)$ is denoted F_j; the lower bound and upper bound of the support of F_j are denoted respectively by p_j^- and p_j^+.[12] With these notations in hand, we now establish a set of lemmata which allows us to identify an upper bound for the entrant's equilibrium payoffs in pricing subgames.

Lemma 5 *Let $k < g(s)$ and $s < 1$. In equilibrium of $G(k,s)$, $p_i^+ \leq \frac{1-ks}{2}$ and $p_e^+ \leq BR_e\left(\frac{1-ks}{2}\right)$.*

Proof: We proceed by iteration; Observe firstly that $p_i^+ \leq \frac{1}{2}$ (the monopoly price) because at any $p_i > \frac{1}{2}$, $\pi_i(p_i, p_e)$ is decreasing in p_i, thus the average $\pi_i(p_i, F_e)$ is also decreasing in

[12] W.l.o.g. pure (price) strategies belong to the compact $[0;1]$ since prices are positive and bounded by the maximal WTP. A mixed strategy is $F \in \Delta$, the space of (Borel) probability measures over $[0;1]$, its support $\Gamma(F)$ is the set of all points for which every open neighborhood has positive measure. We then have $p^- = \inf(\Gamma(F))$ and $p^+ = \sup(\Gamma(F))$.

p_i which proves that such a price cannot belong to the support of F_i. Next, since $BR_e(p_i)$ is increasing and $p_i^+ \leq \frac{1}{2}$, $BR_e(\frac{1}{2})$ is the largest best reply for the entrant to consider. This means that for $p_e > BR_e(\frac{1}{2})$, $\pi_e(p_e, p_i)$ is decreasing in p_e whatever $p_i \leq \frac{1}{2}$, thus the average $\pi_e(p_e, F_i)$ is also decreasing in p_e which proves that $p_e^+ \leq BR_e(\frac{1}{2})$.

One then observes that because $BR_i(p_e)$ for $p_e > \hat{p}_e$ and $BR_e(p_i)$ are both increasing, they cannot cross anymore. Reiterating the previous reasoning, we can sequentially reduce the upper price played by each firm in a Nash equilibrium. This iteration comes to an end at $\bar{p}_i = \frac{1-ks}{2}$ which defines a local maximum, which is independent of p_e. There is no reason to exclude the incumbent from putting mass on that price in equilibrium. We thus end up with $p_i^+ \leq \frac{1-ks}{2}$ and therefore $p_e^+ \leq BR_e\left(\frac{1-ks}{2}\right)$. ■

Lemma 6 *Let $k < g(s)$. In equilibrium of $G(k,s)$, $p_i^+ = \frac{1-ks}{2}$ and the incumbent payoff is the minimax $\overline{\pi}_i$.*

Proof We may check by algebra that when $k < g(s)$, it is true that $2k(1-s) < \bar{p}_i = \frac{1-ks}{2}$. This implies that $BR_e(\bar{p}_i) = \rho_e$ and by the previous lemma, that $p_e^+ \leq \rho_e$. Hence, for p_i in a neighborhood of $\bar{p}_i$, the incumbent's sales are the residual ones D_i^r so that we have $\pi_i(p_i, F_e) = p_i(1 - ks - p_i)$.

If $2k(1-s) \leq p_i^+ < \bar{p}_i$, then $\pi_i(p_i, F_e)$ is strictly increasing over $\left]p_i^+; \bar{p}_i\right[$ which implies that p_i^+ cannot be part of an equilibrium strategy for the incumbent.

If, on the contrary, $p_i^+ < 2k(1-s)$, then the previous argument does not apply because the incumbent's sales might vary. However, if this case occurs then the entrant's demand, when facing F_i, is always of the duopolistic kind without capacity constraint, hence his best reply is the pure strategy ϕ_e computed at the average of p_i. Since the pure strategy equilibrium does not exist over the present domain, the incumbent must be playing a mixed strategy and the only candidate when the entrant plays a pure strategy involves playing the security price $\bar{p}_i$, a contradiction with $p_i^+ < \bar{p}_i$.

We have thus shown that $p_i^+ = \frac{1-ks}{2}$ and since the equilibrium payoff can be computed at any price in the support of F_i, we have $\pi_i(p_i^+, F_e) = p_i^+(1 - ks - p_i^+) = \frac{(1-ks)^2}{4} = \overline{\pi}_i$. ■

Lemma 7 *Let $k < g(s)$. In equilibrium of $G(k,s)$, $p_e^- \leq \hat{p}_e$ if $k \geq h(s)$ and $p_e^- \leq \tilde{p}_e$ if $k \leq h(s)$. The entrant's equilibrium payoff is bounded from above by $k\hat{p}_e(s,k)$ if $k \geq h(s)$ and by $k\tilde{p}_e(s,k)$ if $k \leq h(s)$.*

Proof: Let us consider first the case $k < h(s)$. If $p_e^- > \tilde{p}_e$ then for any $p_i < \frac{p_e^-}{s}$, the incumbent's demand is the monopoly demand whenever $p_e \geq p_e^-$. Hence, $\pi_i(p_i, F_e) = p_i(1 - p_i)$ is strictly increasing, which means the lowest price of the mixed strategy F_i cannot belong to this area. We have thus shown that $p_i^- \geq \frac{p_e^-}{s}$ holds true. If $p_i^- = \frac{p_e^-}{s}$, then at p_i^-, the incumbent is a monopoly whenever $p_e \geq p_e^-$, thus $\pi_i(p_i^-, F_e) = p_i^-(1 - p_i^-) = \frac{p_e^-}{s}\left(1 - \frac{p_e^-}{s}\right) > \frac{\tilde{p}_e}{s}\left(1 - \frac{\tilde{p}_e}{s}\right) = \frac{(1-ks)^2}{4} = \overline{\pi}_i$ by definition of $\tilde{p}_e$ and by the previous lemma. This inequality is a contradiction with p_i^- being in the support of F_i. The last case is thus $p_i^- > \frac{p_e^-}{s}$. Then, $\pi_i(p_i^-, F_e) \geq \pi_i\left(\frac{p_e^-}{s}, F_e\right)$ since p_i^- is an optimal price and $\frac{p_e^-}{s}$ is not; observing that $\pi_i\left(\frac{p_e^-}{s}, F_e\right) = \frac{p_e^-}{s}\left(1 - \frac{p_e^-}{s}\right)$, the previous argument applies and we obtain again a contradiction. This proves that $p_e^- > \tilde{p}_e$ is not true, making true our first claim.

The second claim is then a simple consequence of the fact that the equilibrium payoff can be computed at any price in the support of F_e, hence

$$\pi_e(p_e^-, F_i) = p_e^- \int S_e(p_e^-, p_i) dF_i(p_i) \leq kp_e^- \leq k\tilde{p}_e$$

since sales are bounded by the capacity. The case for $k \geq h(s)$ is identical since the benchmarks $\tilde{p}_e$ and $\hat{p}_e$ play a symmetric role. ■

References

Benassy, J. P. (1989). Market size and substitutability in imperfect competition: a bertrand-edgeworth-chamberlain model. *Review of Economic Studies*, 56:217–234.

Boccard, N. and Wauthy, X. (2010a). Ensuring quality provision through capacity regulation under price competition. *B.E. Journal of Theoretical Economics*, 1.

Boccard, N. and Wauthy, X. (2010b). Equilibrium vertical differentiation in a bertrand model with capacity precommitment. *International Journal of Industrial Organization*, 28(3):288–297.

Canoy, M. (1996). Product differentiation in a bertrand-edgeworth duopoly. *Journal of Economic Theory*.

Choi, C. J. and Shin, H. S. (1992). A comment on a model of vertical product differentiation. *Journal of Industrial Economics*, 40(2):229–231.

Edgeworth, F. (1925). *Papers relating to Political Economy, vol. 1*, chapter The Theory of Pure Monopoly. MacMillan, New York.

Gelman, J. and Salop, S. (1983). Judo economics: Capacity limitation and coupon competition. *Bell Journal of Economics*, 14(2):315–325.

Glicksberg, I. L. (1952). A further generalization of the kakutani fixed point theorem, with application to nash equilibrium points. *Proceedings of American Mathematical Society*, 3(1):170–174.

Kreps, D. and Scheinkman, J. (1983). Quantity precommitment and bertrand competition yields cournot outcomes. *Bell Journal of Economics*, 14(2):326–337.

Krishna, K. (1989). Trade restrictions as facilitating practices. *Journal of International Economics*, 26(3-4):251–270.

Levitan, R. and Shubik, M. (1972). Price duopoly and capacity constraints. *International Economic Review*, 13:111–122.

Mussa, M. and Rosen, S. (1978). Monopoly and product quality. *Journal of Economic Theory*, (18):301–317.

Tirole, J. (1988). *The Theory of Industrial Organization*. MIT press.

Wauthy, X. (2014). From bertrand to cournot via kreps and scheinkman: a hazardous journey. Technical Report 2014-16, CORE.