

Article

Understanding the Axioms and Assumptions of Logical Mathematical Systems through Raster Images: Application to the Construction of a Likert Scale

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Abstract: This article presents different artistic raster images as a resource for correcting misconceptions about different laws and assumptions that underlie the propositional systems of binary logic, Łukasiewicz's trivalent logic, Peirce's trivalent logic, Post's n-valent logic, and Black and Zadeh's infinite-valent logic. Recognizing similarities and differences in how images are constructed allows us to deepen, through comparison, the laws of bivalence, non-contradiction, and excluded middle, as well as understanding other multivalent logic assumptions from another perspective, such as their number of truth values. Consequently, the first goal of this article is to illustrate how the use of visualization can be a powerful tool for better understanding some logic systems. To demonstrate the utility of this objective, we illustrate how a deeper understanding of logic systems helps us appreciate the necessity of employing Likert scales based on the logic of Post or Zadeh, which is the second goal of the article.

Keywords: fuzzy logic; bivalent logic; multivalent logic; conceptual image; laws of logic

MSC: 03A05



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1. Introduction

In the bivalent propositional logical system, every declarative sentence must be either true or, if it is not true, it can only be false. This statement is known as the law of bivalence. Other laws of the bivalent propositional logical system are the law of the excluded middle, which declares that all statements are either true or false, with there being no middle option outside of these two, and the law of non-contradiction, which asserts that statements cannot be true and false at the same time [1]. Let us note that both laws are necessary to justify the law of bivalence, since the law of the excluded middle does not rule out the possibility that a statement can be true and false at the same time.

During the last century, numerous logical systems appeared that do not comply with the law of bivalence. Since these new systems accept more than two truth values for declarative statements, they were called multivalent logic systems. Given the recent appearance of these new ideas, some confusion is common even today regarding what laws or assumptions are associated with the construction of these systems, resulting in a general lack of understanding of how to work within a framework based on acceptance of the laws and assumptions in certain multivalent logical systems. Proof of this is found in the methodology for collecting opinions, evaluations or ratings based on questionnaires with predetermined response scales such as Likert-type [2,3]. Today, there appears to be much confusion regarding the type of coherent answers to the different implicit logical systems

following the questions in said questionnaires, and this is due to a lack of understanding regarding the underlying logical principles and assumptions of the context.

The idea proposed in the article by Klein, Eternson, and Morris [4] can be used as a starting point for addressing the initial research gap that the present study aims to pinpoint. The authors, based on the studies of Herzberg, Mausner, and Snyderman [5]; Czepiel, Rosenberg, and Akerele [6]; Leavitt [7]; and Oliver and Westbrook [8], warn that according to the “two-factor theory”, in many situations, there are elements that produce satisfaction (satisfiers) or dissatisfaction (dissatisfiers). Then, citing the work of Yi [9], in which it is shown that satisfiers are separate and distinct from dissatisfiers and that they are unequal and nonlinear in their effect on many other constructs, including motivation, attitudes, preferences, and behaviours, they clearly warn that satisfiers and dissatisfiers are not opposites to each other and carried out a study regarding these two variables independently. Given the widespread use of Likert scales anchored by “strongly disagree–strongly agree” in many articles, there is a clear need to further understand this current methodological issue. This is, therefore, our first aim.

On the other hand, this research can also be included within a line of research that aims to illustrate how use visualization can be a powerful tool for better understanding some logical concepts. The line of research that explores visual arguments is known as proofs without words [10–12]. The images created in this line should help understand mathematical ideas, demonstrations, and arguments. Therefore, a second possible gap that the article tries to contribute to is the lack of images of this type that can help understand the axioms and assumptions of different logical systems.

With this second aim of providing help in such contexts, the present work seeks to fulfil the objective of improving understanding of the logical laws that underlie the different multivalent propositional logical systems, as well as understanding other multivalent logic assumptions, such as their number of truth values. To this end, a type of image, known as raster, will be used, which will allow us to obtain conceptual images [13] of multiple logical propositional systems that will establish conceptual definitions, eliminating possible cognitive conflicts related to them.

To achieve the aforementioned aims, the article is divided into the following sections: following this introduction come two sections on the materials and methods. The Section 2 entitled “Materials and Methods I: From Bivalent Logic to Multivalent Logics” provides a constructive tour of Peirce and Łukasiewicz’s trivalent logic, Post’s n -valent logic and Black and Zadeh’s infinite-valent logic. Next, in the Section 3, “Materials and Methods II: Colouring the Logics”, we present a proposal to work with raster images as an artistic representation for each of the different logical systems mentioned, lending greater meaning to the logical laws of bivalence, the excluded middle and non-contradiction, as well as other suppositions of multivalent logics. The work continues with the Section 4, “Results and Discussion”, in which the type of Likert scale that researchers should use based on the logical axioms implicit in their research is widely discussed. The work finishes with the Section 5, “Conclusions”, which summarizes the article and justifies the artistic part of the works presented, since it is expected that they will form part of a travelling exhibition through various museums.

2. Materials and Methods I: From Bivalent Logic to Multivalent Logics

Until relatively few years ago, our conception of the world was basically governed by a bivalent logic, the logic of 100% truth (or value 1 associated with true) or 0% truth (or value 0 associated with false), despite the fact that over two thousand years ago, Aristotle (*De interpretatione*, chap. IX) warned that this logic could not assign one of these two truth values to some statements in certain contexts. He gave the example that in a context of free will, the statement “tomorrow there will be a naval battle” cannot be true or false when it is pronounced since, then, the future would be completely determined, invalidating the principle of free choice. Trying to resolve this problem, Łukasiewicz [14] presented the first trivalent logic, assigning the truth value of 0.5 to this problematic type

of sentence. A year later, Post [15] would independently create—though from a purely formal mathematical stance—multivalent logics with any finite number of values, thus founding tetravalent or pentavalent logic, for example. Later discoveries have shown that Peirce had already created a trivalent logical system in 1909, based on the idea of accepting a third truth value for statements that are not definitely true or false but are on the border between the two [16]. This type of logic would bear a relationship of similarity with the ideas of Russell [17] regarding the term “vagueness”, since he accepted that all words are undoubtedly attributable in a certain domain but become questionable within a penumbra, outside which they again become not attributable. Black [18] would present the idea that these statements could have a continuous type of graduation. Years later, Zadeh [19] popularized infinitely multivalent logic under the name of fuzzy logic (At present, the term “fuzzy logic” is considered to group together infinite multivalent logical systems in which, in addition to having differences in the way in which the union, intersection or negation of statements are constructed, the acceptable truth values can be considered as a single value from the interval $[0, 1]$, intervals within the interval $[0, 1]$ or even other structures.. For this reason, during the rest of the article, when we talk about fuzzy logic, we will refer only to the logic proposed by Zadeh, which simply accepts that the truth value of a statement is a number in the interval $[0, 1]$ and accepts that the value of the truth of a negation is 1 minus the truth value of the initial statement, or the truth value of the union and intersection of statements is the maximum and minimum of the truth values of both statements, respectively.), or one in which it is accepted that statements can have any truth value determined between 0 and 1 (inclusive).

The logical laws associated with constructing the different logical propositional systems mentioned are as follows:

- (1) The law of bivalence: each declarative sentence with meaning (Requiring that it has meaning stems from Stuart Mill’s observation that, since it has no meaning, the declarative statement “Abracadabra is a second intention” cannot be conceived in binary logic as being either a true or a false statement.) has exactly one truth value, either true or false.
- (2) The law of the excluded middle: every meaningful declarative sentence is true or false, with no third possible option outside of these two.
- (3) The law of non-contradiction: no meaningful declarative sentence can be true and false at the same time.

In the next section, we will gain a better understanding of how each law is defined through the use of raster images and determine other constructive premises that will allow us to delve into the differences between the different logical systems mentioned above.

3. Materials and Methods II: Colouring the Logics

Raster images, also called bitmaps, present a rectangular grid of pixels suitable for working with the logical laws of bivalence, non-contradiction and the excluded middle, as well as other implicit assumptions of logical systems, such as the assumption that truth values do not change with time or that there is a fixed a priori number of possible truth values.

The essential idea is to associate truth values with colours that represent those values. Thus, the truth values of six statements in a bivalent propositional logical system can be viewed as a raster image in the style of Figure 1, whose pixels are coloured white if the statement is true or black if it is false. Dichotomous thinking, also known as black-and-white thinking, is when your thought patterns assign people, things, and actions to one of two categories. The choice of black and white colours is simply justified by the popularity of this term.

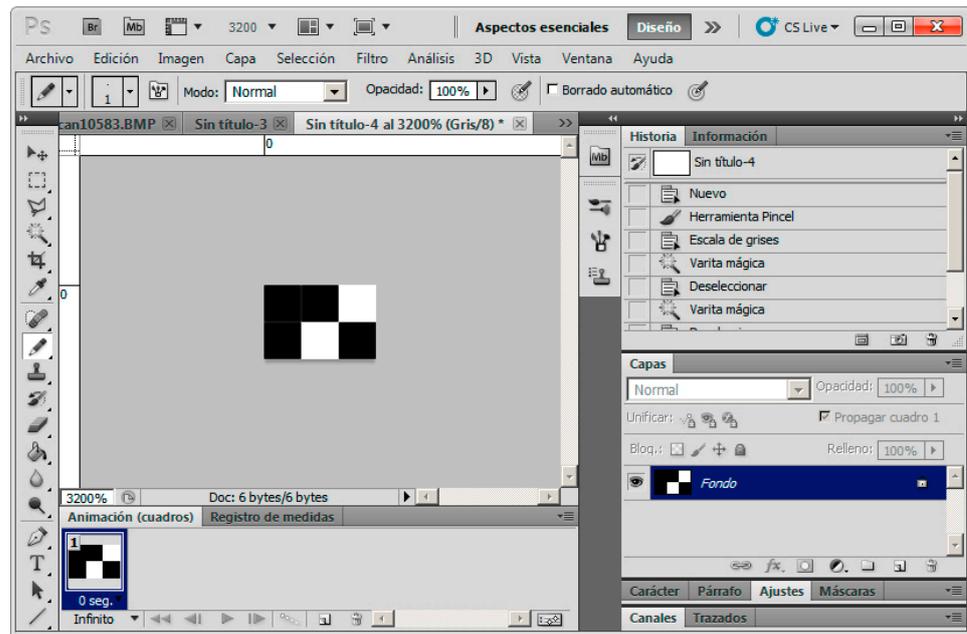


Figure 1. Two-pixels-high and three-pixels-wide raster image, displayed in ADOBE photoshop.

We can establish a rule for the idea by associating the colour of the pixel of the raster image to its truth value based on the following expression:

$$pixel\ color\ P_{ij} = \begin{cases} white\ if\ the\ statement\ (i - 1) * 3 + j\ is\ true \\ black\ if\ the\ statement\ (i - 1) * 3 + j\ is\ false \end{cases}$$

where $1 \leq i \leq 2$ (2 is the number of rows), $1 \leq j \leq 3$ (3 the number of columns) and P_{ij} the pixel associated with the position ij , according to the typical nomenclature used for matrices, where the rows are read from left to right and the columns from top to bottom.

Let us now consider a simple example to understand the rule. Given the following six statements:

- A1: Object A is a square.
- A2: Object B is a square.
- A3: Object C is a square.
- A4: Object D is a square.
- A5: Object E is a square.
- A6: Object F is a square.

According to the rule shown, from the image in Figure 1, it follows that:

- Given that the pixel P_{11} is black, the statement $A_{(1-1).3+1} = A_1$ is false.
- Given that the pixel P_{12} is black, the statement $A_{(1-1).3+2} = A_2$ is false.
- Given that the pixel P_{13} is white, the statement $A_{(1-1).3+3} = A_3$ is true.
- Given that the pixel P_{21} is black, the statement $A_{(2-1).3+1} = A_4$ is false.
- Given that the pixel P_{22} is white, the statement $A_{(2-1).3+2} = A_5$ is true.
- Given that the pixel P_{23} is black, the statement $A_{(2-1).3+3} = A_6$ is false.

This idea can be generalized to raster images of size $n \times m$ (number of rows \times number of columns) according to the following rule:

$$pixel\ color\ P_{ij} = \begin{cases} white\ if\ the\ statement\ (i - 1) * m + j\ is\ true \\ black\ if\ the\ statement\ (i - 1) * m + j\ is\ false \end{cases} \tag{1}$$

where $1 \leq i \leq n, 1 \leq j \leq m$.

Thus, by way of example, Figure 2 provides a quick visualization of the truth values of a group of 3290 (47×70) statements. In this image, the shape serves a purely aesthetic

purpose, and we do not intend for viewers to seek any deeper explanation. Our primary aim is to captivate the observer's attention, encouraging them to reflect on how the choice of colours contributes to the overall discourse.

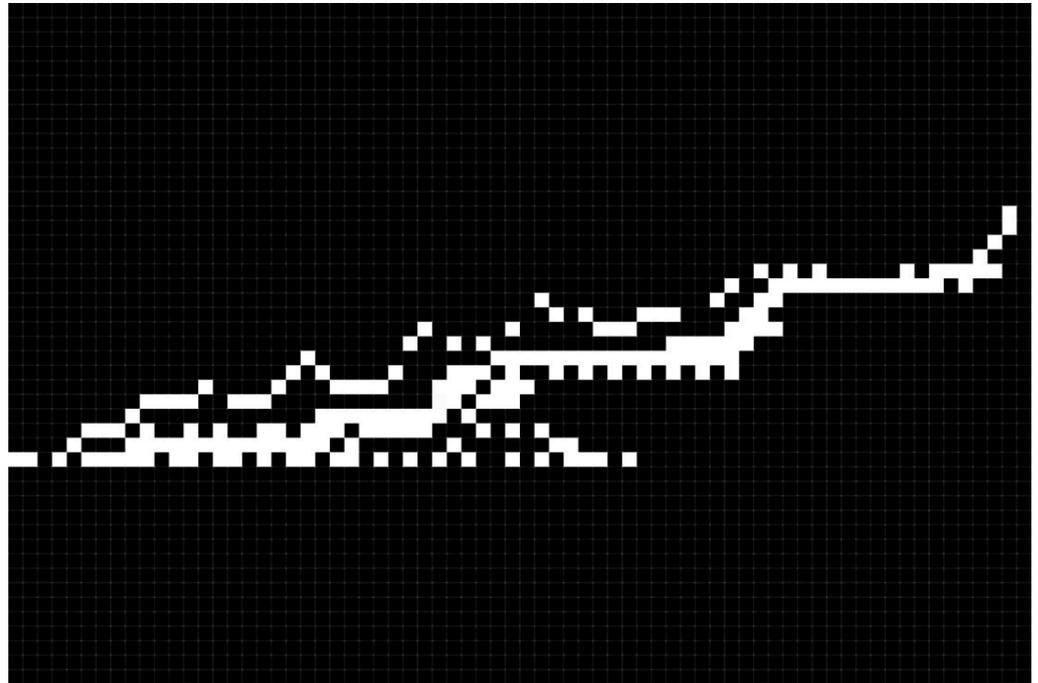


Figure 2. Artistic raster image representative of a bivalent propositional system.

We can observe how the colours in Figure 2 complement the textual information provided in the definition of the laws. They enable us to vividly visualize how, in bivalent or binary propositional logic, the law of the excluded middle is always fulfilled. This is because, by analogy, we can only paint the pixels either black or white, with no third option available. Similarly, we see that the law of non-contradiction is fulfilled, since every pixel, by the very definition of a pixel, can only be painted in a single colour and cannot therefore be painted black and white at the same time. Finally, the law of bivalence is also better understood, since each pixel must necessarily be painted one colour or the other.

Now we have seen that the representation of the bivalent system by raster images created from black and white colours makes sense, the next step is to represent Łukasiewicz's trivalent system using three colours. Since Łukasiewicz's system supposes accepting that contingent future sentences cannot be classified as true or false, it is proposed that a third colour be used, completely different from black or white, so that there is no confusion between them. Therefore, we will not be using any shades of grey. The artist chose the colour green for this purpose. Figure 3 shows a representation of a set of 3290 statements in this system through the use of white, black and green. Let us remember that these artistic images serve a purely aesthetic purpose, and there is no need to seek meaning in the pattern of the shapes. The figure shows how there is now a third truth value different from true and false, represented by the colour green, revealing that the law of the excluded middle is no longer fulfilled. Given that the law of non-contradiction continues to be fulfilled, since no pixel can have two colours at the same time, the image allows us to discuss the possible independent nature of these two laws.

The next logical propositional system that we are going to look at is Peirce's trivalent propositional system, which is documented in his notebook [16] on logic and accepts that the elements that fulfil a predicate can fulfil it in a true or false way, or its degree of compliance is on the boundary between truth and falsehood, coinciding with Russell's [17] idea of vagueness. As an example, we have the sentence: "the car is red". In the words of Russell [17], since the colours constitute a continuum, there will be shades of red that we

will not be able to identify as belonging to that tone, not because we are ignorant of the meaning of the word red, but because colours are words whose extension of application is essentially doubtful. Thus, there will be cars with a shade of red that we will not be able to identify with total certainty as a red car, because we will have doubts between the being and not being of that specific quality. Consequently, there will be situations in which the sentence “the car is red” can be seen as both a true and a false sentence.

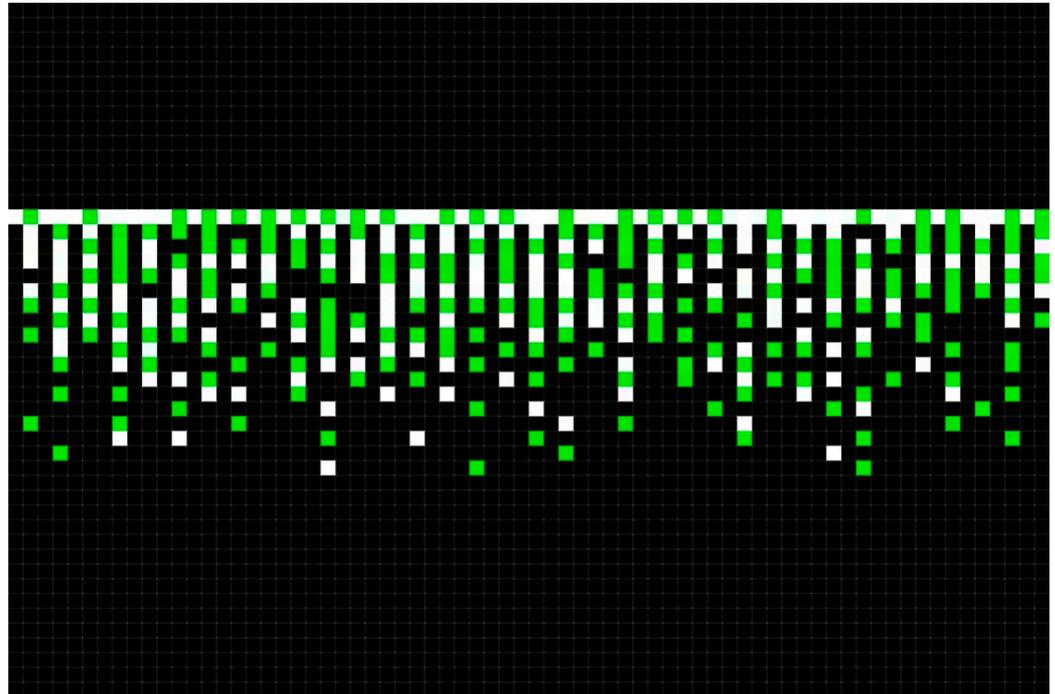


Figure 3. Artistic raster image representative of Łukasiewicz's trivalent propositional system.

To work on this type of logic, a raster image in white, black, and a shade of grey can be used to represent a logic system that supports the truth values *true*, *false* and *both true and false at the same time*. Peirce's trivalent logic does not fulfil any of the three laws listed, not even the law of non-contradiction, since we have statements that can be considered true and false at the same time. Figure 4 presents an artistic image that helps us understand the non-compliance of the three laws given this logic.

Figure 5 is an artistic raster image associated with Post's 4-valent logic, which is an extension of Peirce's trivalent logic system. Specifically, the figure presents a raster image in white, light grey, dark grey and black, representing a logical system that admits four truth values and whose possible expression to assign colour is:

$$\text{pixel colour} = \begin{cases} \text{white} & \text{for absolutely true statements} \\ \text{light grey} & \text{for statements that are more true than false} \\ \text{dark grey} & \text{for statements that are more false than true} \\ \text{black} & \text{for absolutely false statements} \end{cases}$$

We observe that in this tetravalent logical system, none of the three studied laws are fulfilled. The fact that there are more colours than just black and white with which to paint the pixel serves to negate the law of the excluded middle. Similarly, being able to paint a pixel with a mixture of black and white allows the law of non-contradiction to be negated, since it paints the pixel with a colour obtained from black and white at the same time. Finally, the law of bivalence is not fulfilled either, since being able to paint the pixel in colours other than black and white is consistent with the fact that there are meaningful declarative statements that do not have exactly one single truth value, true or false.

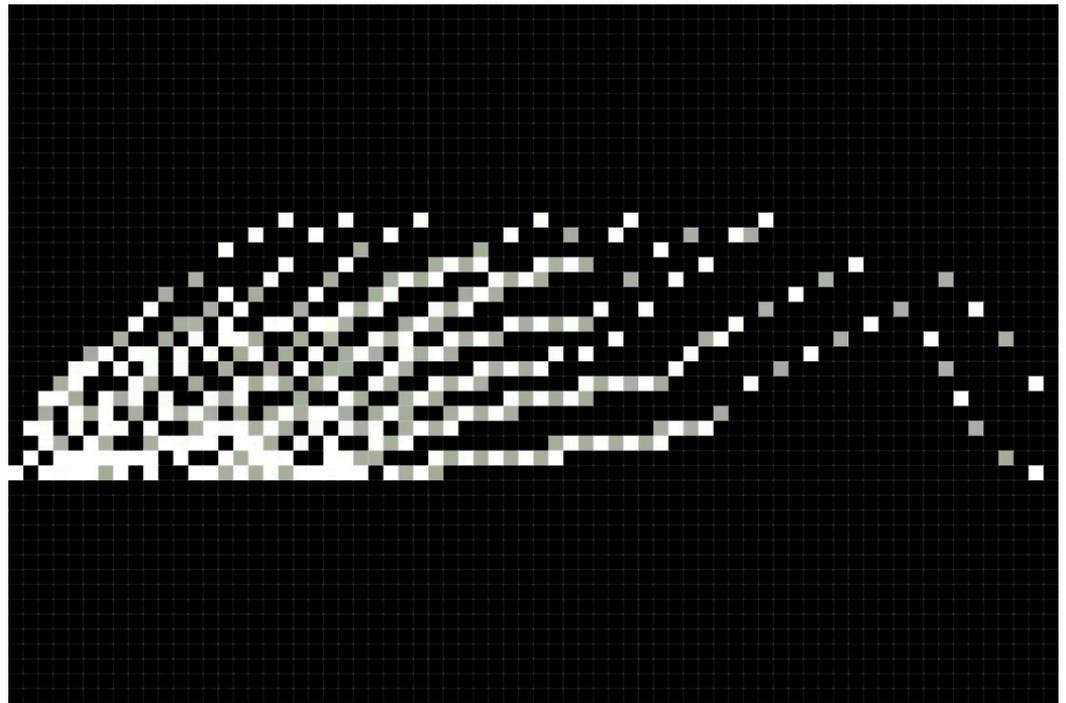


Figure 4. Artistic raster image representative of Peirce's trivalent propositional system.

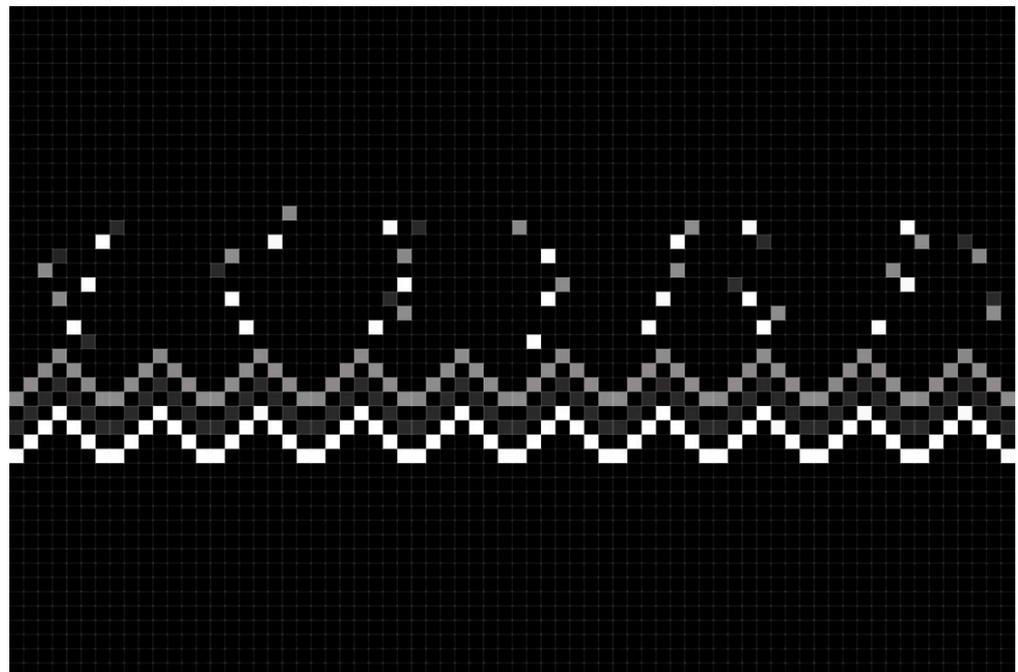


Figure 5. Artistic raster image representative of Post's tetravalent propositional system.

Finally, Figure 6 shows a raster image to represent the logical system that allows each proposition to be assigned a truth value within a continuous gradient of infinite options. For this reason, the pixels can be painted in infinite shades of grey between white and black, inclusive. Since it is not possible to represent infinite shades of grey on a canvas, it was decided to suggest the infinite greys by progressively darkening from white to black in different parts of the work. This logical system, identified with the fuzzy logic system created by Black [18] and Zadeh [19], does not fulfil any of the laws in the list either, and can

be seen as an extreme case when n tends to infinity from Post's n -valent systems derived from Peirce's trivalent.

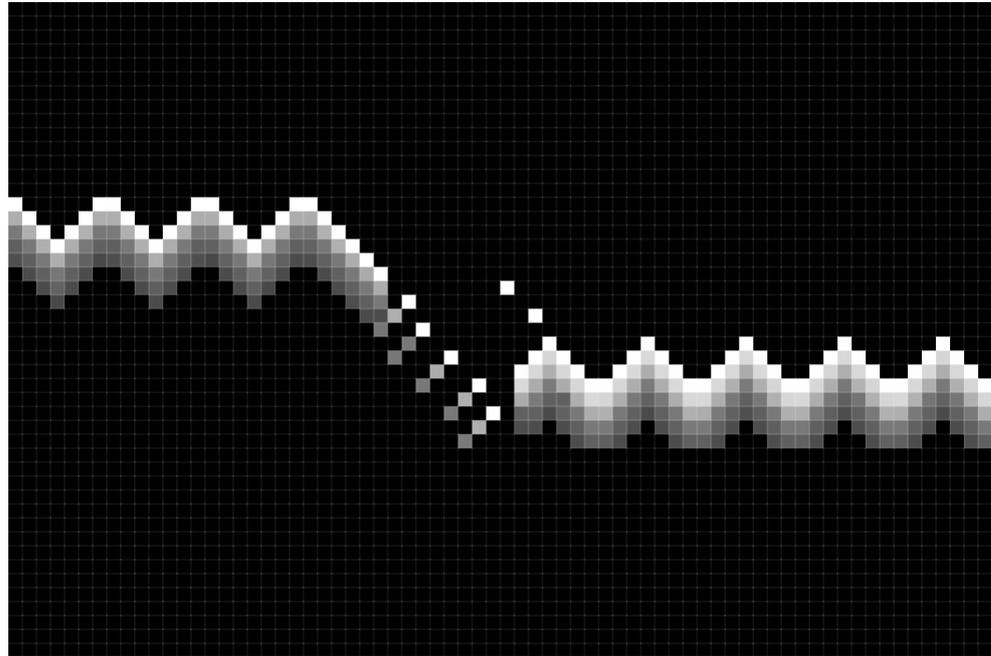


Figure 6. Artistic raster image representative of a propositional infinite-valent system.

Black and Zadeh's logical system currently provides one of the theoretical bases closest to human thought for modelling problems that incorporate uncertainty in the models mentioned [20–23].

4. Results and Discussion

This article presents five raster-type artistic works creating conceptual images to represent binary propositional logic, Łukasiewicz's trivalent propositional logic, Peirce's trivalent propositional logic, Post's n -valent propositional logics, and Black and Zadeh's infinite-valent propositional logic. The similarities and differences in these images provide a deeper understanding of the definitions of the laws of the excluded middle, non-contradiction, bivalence and other assumptions of the mentioned logical systems, such as their number of items or the fact that the truth values of propositions in these systems do not change over time. Consequently, the work shows that, aside from the usual idealistic conception of aesthetic contemplation, art can also lead to other forms of relationship between an artist and the public. More specifically, in the first place, this work evidences a connection between art and mathematics from a formative point of view, facilitating a significant understanding of abstract mathematical ideas, which enrich the reader's knowledge of the laws, axioms and assumptions in multiple logical systems. Secondly, the work presents the construction of different logical systems based on the laws of non-contradiction, the excluded middle and the assumption of the number of truth values in the system. In this regard, it provides a greater understanding of how explicit laws and assumptions comply with various multivalent logical systems, ultimately showing that the acceptance or not of the laws of non-contradiction and the excluded third middle build different multivalent logical systems. Consequently, the work carried out can be classified within the research line related to the debate around the possibilities offered by the acceptance or denial of laws in the field of logic, an open line with continual contributions to this day [24–27], the fruit of which continues to grow on a practical level [28–30].

We would like to note that the artworks presented here will form part of an exhibition on multivalent logics, to go on display shortly. This adheres to the objectives of research in the educational field, which the research group to which the authors belong has been

carrying out in recent years. As a result of an agreement with the Barcelona Science Museum, the authors participated as curators of the exhibition that commemorated the 50th anniversary of the creation of the first article on fuzzy logic [31]. Based on the multiple ideas arising from that exhibition, another exhibition was held on the sorites paradox at the Catalan Museum of Mathematics, involving ten artistic works being presented around this concept [32].

Finally, we aim to utilize the three laws presented to delve deeper into the construction of Likert scales. We will discover that the type of scale for evaluating the quality or characteristic to be measured is related to the type of logical axioms chosen for the statistical study, classifiable under one of the logics presented in this work.

Firstly, the scale type is illustrated in Figure 7, where one must choose a single value from various options. The consumer's overall evaluation of the foreign product is measured using five items on a Likert scale anchored by "strongly disagree-strongly agree". This type of scale has been used and continues to be used in a multitude of experimental works (see, for instance, Shafieizadeh et al. [33] or Kraus et al. [34]).

Please select an option related to the statement below:

I like the idea of buying Catalan products

1 - Strongly disagree

2 - Disagree

3 - Neither agree or disagree

4 - Agree

5- Strongly agree

Figure 7. Likert scale anchored by "strongly disagree-strongly agree".

There are different ways of presenting the same scale. Figures 8 and 9 are two of them.

Please select an option related to the statement below:

I like the idea of buying Catalan products

1 2 3 4 5

Strongly disagree Strongly agree

Figure 8. Second example of Likert scale anchored by "strongly disagree-strongly agree".

If the researcher opts for this approach, they will implicitly be employing the axioms of binary logic.

This arises from the fact that the person being surveyed must select only one item, following the idea that it is either an option that identifies a set, or the answer lies within the complementary set. There is no third option. Since they cannot mark two items simultaneously, we are adhering to the law of non-contradiction, preventing a meaningful

declarative sentence from being true and false simultaneously. By complying with both laws, we automatically satisfy the law of bivalence.

Please select an option related to the statement below:

I like the idea of buying Catalan products



Strongly Disagree Disagree Neutral Agree Strongly Agree

Figure 9. Third example of Likert scale anchored by “strongly disagree-strongly agree”.

The choice of this type of scale should be carefully considered by the researcher. They should be aware that this scale type presents inherent issues related to binary classification. For instance, if there is uncertainty between two correlated options, the researcher will never know. Likewise, if a consumer likes certain aspects of a Catalan product but dislikes others, such as enjoying the taste and price but disliking the use of non-local ingredients, selecting only one option would not fully represent their feelings.

In this sense, something can be both liked and disliked simultaneously. This phenomenon is known in psychoanalysis as ambivalence, a concept introduced by Bleuler [35], referring to a pronounced emotional attitude in which contradictory impulses coexist.

A possible way to try to solve this conflict may be to use a scale like the one presented in Figure 10. This type of Likert scale involves marking a single item on a scale that extends the first type with an option labelled “Don’t know”.

Please select an option related to the statement below:

I like the idea of buying Catalan products

1 - Strongly disagree Z - Don't know

2 - Disagree

3 - Neither agree or disagree

4 - Agree

5 - Strongly agree

Figure 10. Likert scale anchored by “strongly disagree-strongly agree” plus “Don’t Know” option.

Let us observe that in this case, we have a scale that follows the axioms presented in Łukasiewicz’s trivalent logic. If we cannot choose between an item and its complementary set, the response “Don’t Know” must be marked. Therefore, there is a third possible truth value (the law of the excluded middle is not upheld), and by having to select only one, we adhere to the law of non-contradiction since no meaningful declarative sentence can be true and false simultaneously.

With this scale, indeed, similar issues to those mentioned in the previous scale can be identified, although it would not specifically pinpoint the problem that led to choosing the “Don’t Know” option. Consequently, it would be more accurate to present two separate scales to measure both satisfaction and dissatisfaction, where a value of 0 in both represents absence and a value of 1 represents totality. In Figures 11 and 12, two examples of these can be seen.

Select one option from each column related to the following statement:

I like the idea of buying Catalan products

<input type="radio"/> Totally agree	<input type="radio"/> Totally disagree
<input type="radio"/> Partially agree	<input type="radio"/> Partially disagree
<input type="radio"/> Not agree	<input type="radio"/> Not disagree

Figure 11. Likert scale with two columns of three options.

Select one option from each column related to the following statement:

I like the idea of buying Catalan products

<input type="radio"/> Strongly agree	<input type="radio"/> Strongly disagree
<input type="radio"/> Moderately agree	<input type="radio"/> Moderately disagree
<input type="radio"/> Slightly agree	<input type="radio"/> Slightly disagree
<input type="radio"/> Not agree	<input type="radio"/> Not disagree

Figure 12. Likert scale with two columns of four options.

Let us observe that, in this case, the scales follow the axioms presented in Peirce’s trivalent logic and Post’s n-valent logic.

If the researcher prefers, it is possible to use a format with two statements and a single column to indicate the degree of truth for each statement. This format aligns more closely with the implicit logic of the questionnaires while also simplifying Likert scales to a single column. An example of this idea can be seen in Figure 13.

<p style="text-align: center;">I like the idea of buying Catalan products</p> <p>Please select an option related to the statement below:</p> <ul style="list-style-type: none"> <input type="radio"/> I agree 100% <input type="radio"/> I agree 75% <input type="radio"/> I agree 50% <input type="radio"/> I agree 25% <input type="radio"/> I agree 0% 	<p style="text-align: center;">I dislike the idea of buying Catalan products</p> <p>Please select an option related to the statement below:</p> <ul style="list-style-type: none"> <input type="radio"/> I agree 100% <input type="radio"/> I agree 75% <input type="radio"/> I agree 50% <input type="radio"/> I agree 25% <input type="radio"/> I agree 0%
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Figure 13. Likert scale in a 5-valued Post context.

It should be noted that in some research, Likert scales within the framework of fuzzy axioms have been used. In these cases, researchers (see, for instance, Lalla et al. [36], Bharadwaj [37], Lazim and Osman [38]) propose assigning a truth value to each statement on the scale, indirectly allowing for the marking of different items. Figure 14 shows an example of question presentation in this type of proposal.

<p style="text-align: center;">I like the idea of buying Catalan products</p> <p style="text-align: center;">Please, assign a truth value to each statement on the scale.</p> <p>I agree 100% <input style="width: 80px;" type="text"/></p> <p>I agree 75% <input style="width: 80px;" type="text"/></p> <p>I agree 50% <input style="width: 80px;" type="text"/></p> <p>I agree 25% <input style="width: 80px;" type="text"/></p> <p>I agree 0% <input style="width: 80px;" type="text"/></p>	<p style="text-align: center;">I dislike the idea of buying Catalan products</p> <p style="text-align: center;">Please, assign a truth value to each statement on the scale.</p> <p>I agree 100% <input style="width: 80px;" type="text"/></p> <p>I agree 75% <input style="width: 80px;" type="text"/></p> <p>I agree 50% <input style="width: 80px;" type="text"/></p> <p>I agree 25% <input style="width: 80px;" type="text"/></p> <p>I agree 0% <input style="width: 80px;" type="text"/></p>
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Figure 14. Question showing the identification of each Likert response category with a fuzzy subset.

Finally, we want to propose an alternative Likert scale within the framework of fuzzy axioms. Figure 15 provides an example that aligns with our proposed idea.

<p style="text-align: center;">I like the idea of buying Catalan products</p> <p style="text-align: center;">Using the color scale, indicate the level of agreement with your opinion, considering the items that contribute to your perspective</p> <div style="display: flex; align-items: center;"> <div style="width: 20px; height: 100px; background: linear-gradient(to bottom, black, white); margin-right: 5px;"></div> <div style="display: flex; flex-direction: column; justify-content: space-between; width: 100%;"> <p>I agree 100%</p> <p>I agree 75%</p> <p>I agree 50%</p> <p>I agree 25%</p> <p>I agree 0%</p> </div> </div>	<p style="text-align: center;">I dislike the idea of buying Catalan products</p> <p style="text-align: center;">Using the color scale, indicate the level of disagreement with your opinion, considering the items that contribute to your perspective</p> <div style="display: flex; align-items: center;"> <div style="width: 20px; height: 100px; background: linear-gradient(to bottom, black, white); margin-right: 5px;"></div> <div style="display: flex; flex-direction: column; justify-content: space-between; width: 100%;"> <p>I agree 100%</p> <p>I agree 75%</p> <p>I agree 50%</p> <p>I agree 25%</p> <p>I agree 0%</p> </div> </div>
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Figure 15. Question showing the identification of each Likert response category with a fuzzy subset.

The grayscale representation in this Figure serves as a reminder that values in fuzzy logic range from 0 to 1 (inclusive). It is worth noting that interviewees do not have to mark a single value on the scale; they can select a range of values to indicate the interval in which they believe the true value lies.

We conclude by noting that these scales associated with fuzzy thinking do not fulfil the law of the excluded middle, as they encompass a broader range of possibilities beyond mere truth and falsehood values. Additionally, they do not adhere to the law of non-contradiction, as they permit the possibility of something being both true and false at the same time.

5. Conclusions

We can conclude that the two aims proposed at the beginning of the article have finally been achieved. First of all, it was shown how raster images can serve as valuable tools for elucidating complex concepts and their associated principles to a general audience. They were instrumental in verifying the following facts:

- (1) In bivalent or binary classical propositional logic, the law of the excluded middle is always fulfilled since, by analogy, we can only paint the pixel black or white and there

is no third option. Similarly, the law of non-contradiction is fulfilled, since every pixel, by the very definition of a pixel, can only be painted in a single colour and cannot, therefore, be painted black and white at the same time. Finally, the law of bivalence is fulfilled since each pixel must necessarily be painted in one colour or the other.

- (2) In Łukasiewicz's trivalent logic, the law of the excluded middle is not fulfilled since, by analogy, we can paint the pixel black, white or green and, thus, there is a third option. On the other hand, the law of non-contradiction is fulfilled, since every pixel, by the very definition of a pixel, can only be painted in a single colour and cannot therefore be painted by two colours at the same time. Finally, the law of bivalence is not fulfilled since each pixel can be painted in three colours.
- (3) In Peirce's trivalent logic, the law of the excluded middle is not fulfilled since, by analogy, we can paint the pixel black, white or by a combination of black and white and there is a third option. In addition, the law of non-contradiction is not fulfilled since every pixel can always be painted by a combination of black and white at the same time. Finally, the law of bivalence is not fulfilled since each pixel can be painted in three colours.
- (4) In Post's 4-valent logic, which is an extension of Peirce's trivalent logic system, none of the three studied laws are fulfilled.
- (5) In Black and Zadeh's infinite-valent logic, which is an extension of Peirce's trivalent logic system, none of the three studied laws are fulfilled.

As observed in the previous section, mastering the axiomatics of diverse system logics enhances proficiency in utilizing associated tools, such as selecting appropriate Likert scales for specific research, which was the second objective of this investigation.

To conclude, it is important to highlight that the use of raster images as a conceptual representation of law definitions, following the research approach presented by Tall and Vinner [13], is not the only option. For instance, raster images can be employed to explore the complementary, union, and intersection of various logical systems. While Zadeh primarily used the complement with respect to 1, the maximum, and the minimum [19], there exist endless possibilities for exploration. Moreover, this approach also provides the opportunity to investigate when two logical systems are isomorphic. The ability to confirm this fact at a glance can be exceptionally valuable. Consequently, this work opens up new avenues for research, much like the research direction that Nelsen [10] and others [11,12] are currently pursuing in the realm of theorem proving through images.

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References

1. DeVidi, D.; Solomon, G. On Confusions About Bivalence and Excluded Middle. *Dialogue* **1999**, *38*, 785. [\[CrossRef\]](#)
2. Likert, R. A technique for the measurement of attitudes. *Arch. Psychol.* **1932**, *22*, 55.
3. Likert, R.; Roslow, S.; Murphy, G. A simple and reliable method of scoring the Thurstone attitude scales. *J. Soc. Psychol.* **1934**, *5*, 228–238. [\[CrossRef\]](#)
4. Klein, J.G.; Ettenson, R.; Morris, M.D. The animosity model of foreign product purchase: An empirical test in the People's Republic of China. *J. Mark.* **1998**, *62*, 89–100. [\[CrossRef\]](#)
5. Herzberg, F.; Mausner, B.; Snyderman, B.B. *The Motivation to Work*, 2nd ed.; John Wiley Sons: New York, NY, USA, 1959.
6. Czepiel, J.A.; Rosenberg, L.J.; Akerele, A. *Perspectives on Consumer Satisfaction*. Quoted in *AMA Educators*; American Marketing Association: Chicago, IL, USA, 1974; p. 122.

7. Leavitt, C. Consumer satisfaction and dissatisfaction: Bipolar or independent. In *Conceptualization and Measurement of Consumer Satisfaction and Dissatisfaction*; Marketing Science Institute: Cambridge, MA, USA, 1977; pp. 132–149.
8. Oliver, R.L.; Westbrook, R.A. The factor structure of satisfaction and related postpurchase measures. In *New Findings on Consumer Satisfaction and Complaining*; Indiana University Press: Bloomington, IN, USA, 1982; pp. 11–14.
9. Yi, Y. A critical review of consumer satisfaction. *Rev. Mark.* **1990**, *4*, 68–123.
10. Nelsen, R.B. *Proofs without Words: Exercises in Visual Thinking*; MAA Service Center: Washington, DC, USA, 1993.
11. Allwein, G.; Barwise, J. (Eds.) *Logical Reasoning with Diagrams*; Oxford University Press: Oxford, UK, 1996.
12. Alsina, C.; Nelsen, R.B. Proof Without Words: Ptolemy's Inequality. *Math. Mag.* **2014**, *87*, 291. [[CrossRef](#)]
13. Tall, D.; Vinner, S. Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educ. Stud. Math.* **1981**, *12*, 151–169. [[CrossRef](#)]
14. Łukasiewicz, J. O logice trójwartościowej. *Ruch Filozoficzny.* **1920**, *5*, 170–171.
15. Post, E.L. Introduction to a general theory of elementary propositions. *Am. J. Math.* **1921**, *43*, 163–185. [[CrossRef](#)]
16. Fisch, M.; Turquette, A. Peirce's triadic logic. In *Transactions of the Charles S. Peirce Society*; Indiana University Press: Bloomington, IN, USA, 1966; pp. 71–85.
17. Russell, B. Vagueness. *Australas. J. Psychol. Philos.* **1923**, *1*, 84–92. [[CrossRef](#)]
18. Black, M. Vagueness: An exercise in logical analysis. *Philos. Sci.* **1937**, *4*, 427–455. [[CrossRef](#)]
19. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [[CrossRef](#)]
20. Ferrer-Comalat, J.C.; Linares-Mustarós, S.; Rigall-Torrent, R. Incorporating Fuzzy Logic in Harrod's Economic Growth Model. *Mathematics* **2021**, *9*, 2194. [[CrossRef](#)]
21. Ferrer-Comalat, J.C.; Corominas-Coll, D.; Linares-Mustarós, S. A Fuzzy Economic Dynamic Model. *Mathematics* **2021**, *9*, 826. [[CrossRef](#)]
22. Linares-Mustarós, S.; Ferrer-Comalat, J.C.; Corominas-Coll, D.; Merigó, J.M. The weighted average multiexperton. *Inf. Sci.* **2021**, *557*, 355–372. [[CrossRef](#)]
23. Zeng, S.; Zhou, J.; Zhang, C.; Merigó, J.M. Intuitionistic fuzzy social network hybrid MCDM model for an assessment of digital reforms of manufacturing industry in China. *Technol. Forecast. Soc. Chang.* **2022**, *176*, 121435. [[CrossRef](#)]
24. Day, T.J. Excluded middle and bivalence. *Erkenntnis* **1992**, *37*, 93–97. [[CrossRef](#)]
25. Doull, F. The principle of excluded middle then and now: Aristotle and principia mathematica. *Animus* **1996**, *1*, 53–66.
26. Barukčić, I. General contradiction law. *Causation* **2006**, *1*, 5–31.
27. Tennant, N. Does Choice Really Imply Excluded Middle? Part II: Historical, Philosophical, and Foundational Reflections on the Goodman–Myhill Result. *Philos. Math.* **2020**, *29*, 28–63. [[CrossRef](#)]
28. Merigó, J.M.; Gil-Lafuente, A.M.; Yager, R.R. An overview of fuzzy research with bibliometric indicators. *Appl. Soft Comput.* **2015**, *27*, 420–433. [[CrossRef](#)]
29. Alfaro-García, V.G.; Merigó, J.M.; Alfaro Calderón, G.G.; Plata-Pérez, L.; Gil-Lafuente, A.M.; Herrera-Viedma, E. A citation analysis of fuzzy research by universities and countries. *J. Intell. Fuzzy Syst.* **2020**, *38*, 5355–5367. [[CrossRef](#)]
30. López-Guaque, J.A.; Gil-Lafuente, A.M. Fifty years of fuzzy research: A bibliometric analysis and a long-term comparative overview. *J. Intell. Fuzzy Syst.* **2020**, *38*, 5413–5425. [[CrossRef](#)]
31. Linares-Mustarós, S.; Ferrer-Comalat, J.C.; Bertran-Roura, X.; Corominas-Coll, D.C.; Hidalgo-Gil, J. *50 Years of Fuzzy Ideas: Catalogue for the Exhibition "Explore", Held at CosmoCaixa in Barcelona to Commemorate Half a Century of Fuzzy Logic (13–21 November 2015)*; University of Girona, Publications Service: Girona, Spain, 2018.
32. Viladevall, Q. *From Everything to Nothing. Exhibition Catalogue*; University of Girona, Publications Service: Girona, Spain, 2021.
33. Shafieizadeh, K.; Alotaibi, S.; Tao, C.W.W. How do authenticity and quality perceptions affect dining experiences and recommendations of food trucks? The moderating role of perceived risk. *Int. J. Hosp. Manag.* **2021**, *93*, 102800. [[CrossRef](#)]
34. Kraus, S.; Sen, S.; Savitskie, K.; Kumar, S.K.; Brooks, J., Jr. Attracting the millennial customer: The case of food trucks. *Br. Food J.* **2022**, *124*, 165–182. [[CrossRef](#)]
35. Bleuner, E. The fundamental symptoms. In *Dementia Praecox or the Group of Schizophrenias*; Bleuler, E., Ed.; Ziskin, J., Translator; International University Press: New York, NY, USA, 1950; pp. 14–54.
36. Lalla, M.; Facchinetti, G.; Mastroleo, G. Ordinal scales and fuzzy set systems to measure agreement: An application to the evaluation of teaching activity. *Qual. Quant.* **2004**, *38*, 577–601. [[CrossRef](#)]
37. Bharadwaj, B. *Development of a Fuzzy Likert Scale for the WHO ICF to include Categorical Definitions on the Basis of a Continuum*; Wayne State University: Detroit, MI, USA, 2007.
38. Lazim, M.A.; Osman, M.T.A. Measuring teachers' beliefs about Mathematics: A fuzzy set approach. *Int. J. Soc. Sci.* **2009**, *4*, 39–43.

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