# Restricted environments and incentive compatibility in interdependent values models[1]

by

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Abstract: We study mechanisms that operate in interdependent values environments. We show that when defined on knit and strict environments, only constant mechanisms can be ex post incentive compatible. Knitness is also necessary for this result to hold for mechanisms with two alternatives in the range. For partially knit and strict environments, we prove that ex post incentive compatibility extends to groups, and that strategy-proofness implies strong group strategy-proofness in the special case of private values. The results extend to mechanisms operating on non-strict domains under an additional requirement of respectfulness. We discuss examples of environments where our theorems apply.

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# 1 Introduction

A major concern when designing economic mechanisms is to provide agents with incentives to reveal their true characteristics. Setting aside some obviously unsatisfactory solutions, it is well understood that attaining this objective is not always possible. Moreover, when it is, a conflict often arises between the mechanisms' incentive compatibility and other desirable properties. These generic statements hold for different formulations of the mechanism design problem, and for various concepts of equilibrium. Hence, a mechanism can only meet attractive lists of desiderata if the class of problems to be dealt with is somewhat constrained.

In social choice theory, where mechanisms are defined as functions whose domains are subsets of preference profiles, these constraints on the relevant situations to be considered are called domain restrictions, in contrast to the notion of universal domain that was the basis of fundamental theorems like Arrow's or Gibbard and Satterthwaite's. The term is not always explicitly used in the larger literature on mechanism design, where assumptions on what economic situations lie within the scope of each model are usually predicated directly on the structure of the set of alternatives, or on the types of agents. At any rate, we think it is interesting to explore the consequences of the modeler's choice of a domain for a mechanism upon the properties that one may expect it to satisfy.

Our purpose in the present paper is to study such consequences under two different sorts of domains for mechanisms operating in general situations, including those where the agent's values are interdependent. Our choice of domains results from an attempt to capture, in a unifying spirit, essential features of different models that can be found in the literature and lead to similar results in spite of their diverse premises.

Before we elaborate on our present endeavor, it will be useful to refer to our previous and parallel work regarding mechanism design in private values environments (Barberà, Berga, and Moreno, 2016). There, we identified numerous models and domains under which strategy-proofness was compatible with other desirable properties. Yet, we also observed that, under specific circumstances, one may define mechanisms that are not only individually strategy-proof but also (weakly) group strategy-proof. Then, we studied the characteristics of domains that are common to those models leading to negative results, and also those of domains admitting non-trivial group strategy-proof mechanisms in a diversity of setups.

In the same unifying spirit as in that earlier work, we consider here the general case where agent's types may be interdependent, while still keeping an eye on the particular case of private values. Our starting point toward a general model starts from the observation that, in contexts where values are interdependent, the incentives provided by a mechanism not only depend on the type profiles in its domain, but also on the properties of the preference function associating a profile of agent's preferences to each one of types. We define an environment as a pair formed by the set of admissible type profiles and an associated preference function, and argue that what matters to determine the properties of mechanisms defined on a given family of types depends on that family and also on those preference profiles that are induced for each profile of types through the associated preference function. Restrictions on preference domains are a particular case of our general framework for the case of private values.

Within this larger context we shift our attention to the notions of ex post individual and group incentive compatibility, which are natural counterparts, in the case of interdependent values, to our previous focus on individual and group strategy proofness.

Ex post incentive compatibility is an attractive and well-studied requirement guaranteeing truthful revelation of types to be a Nash equilibrium in all the games that result from any specification of possible type profiles.[1] This equilibrium concept guarantees belief-free implementation (often called "robust implementation"), and it is equivalent to it for social choice functions (Corollary 1 in Bergemann and Morris, 2005). For private values, ex post incentive compatibility is equivalent to strategy-proofness. We also introduce a second concept, that of ex post group incentive compatibility, under which truthful revelation is required to be a strong Nash equilibrium, in order to capture additional features related to the possibility of coordinated action by several individuals. These are our main target properties, and the only ones we need to obtain our main results for environment where the preferences of agents are always strict.

We look for characteristics of domains leading to the same strong impossibility result that appears in different strands of the literature, whereby essentially only constant mechanisms can be ex post incentive compatible (see for example, Austen-Smith and Feddersen, 2006, Che, Kim, and Kojima, 2015, Dasgupta and Maskin, 2000, Jehiel, Meyer-Ter-Vehn, Moldovanu, and Zame, 2006).[2] After careful examination of different models that reach the same conclusion from different starting points, we propose a condition on environments that we call knitness. Our first result proves that only constant mechanisms can be ex post incentive compatible in environments where the preferences of agents are strict. If agents may be indifferent among several alternatives, we need to use an additional condition on mechanisms that we call respectfulness (see Theorem 1 and Corollary 1). This condition, when applied to private values is a relative of non-bossiness (Satterthwaite and Sonnenschein, 1981), but less demanding than this or other similar ones analyzed in Thomson (2016). It essentially rules out manipulations by one agent that could affect others while not gaining anything in exchange, thus opening the way to bribes.

Then, in an exercise that parallels our previous work for the case of private values, we define a notion of ex post group incentive compatibility and discuss its connection with standard (individual) ex post incentive compatibility. We do that for environments satisfying a second condition that we call partial knitness that applies to interdependent values and show that respectful ex post incentive compatible mechanisms defined on partially knit environments also satisfy the stronger version of this condition for groups (see Theorem 4 and Corollary 2). Also observe that in private values this result admits a second reading that is that strategy-proofness is equivalent to strong group strategy-proofness (see Corollary 3).

Both knitness and partial knitness are rather abstract conditions and their use requires justification. A major argument showing the importance of knitness is that, as we prove in Theorem 2, it is not only a sufficient but actually a necessary condition on environments

[1] The study of incentive compatibility in Bayesian terms was started by d'Aspremont and Gérard-Varet (1979), and Arrow (1979), and its appropriate formulation and results depend on the information that will be available to the agents at the time where the analysis is carried out. The case of interdependent values was first studied by D'Aspremont, Crémer, and Gérard-Varet (1990). The notion of ex post incentive compatibility corresponds to the time where agents have received all possible information, and can be defined without attributing cardinal utility to agents, as it does not require Bayesian updating. See Jackson (2003).

[2] We devote special attention to Che, Kim, and Kojima (2015) and to Austen-Smith and Feddersen (2006) that inspired us for the applications in Section 6.

to precipitate the negative result that, for mechanisms with binary ranges, only constant ones can be ex post incentive compatible and respectful. In Theorem 3 we show that sufficient violations of knitness allow for non-constant mechanisms to satisfy the same incentives property if the range's size is larger than two. As for partial knitness, it is a condition in a similar vein but weaker than knitness, that can be satisfied by both private and interdependent values environments (see Pourpouneh, Ramezanianz, and Sen, 2019 and our examples in Subsection 6.2. See Propositions 2, 3, 4 for private values environment in Subsection 6.1). This contrasts with knitness, which cannot be satisfied in the case of private values (see Proposition 1).

After this introduction, the paper proceeds as follows. In Section 2 we present the general framework, define the properties of mechanisms and the conditions on environments that we propose and shall concentrate on. Section 3 contains the impossibility result and the necessity of knitness for the result to hold for binary range mechanisms. In addition, we also discuss some necessary characteristics of domains for the existence of non-constant ex post incentive compatible mechanisms with larger ranges. Section 4 presents a sufficient condition on environments to get the equivalence between ex post individual and group incentive compatibility. While in Section 5 we provide a discussion of our two conditions on environments, Section 6 provides examples of applications and ties them in with our general framework and main results. Some comments about further research are presented in Section 7. Finally, Appendix A presents the proofs of the results related to the applications, and Appendices B and C are mainly devoted to throw light on our two conditions on environments. In the former, we provide examples that illustrate their characteristics, clarify their role in different results and show the possibility of finding ex post incentive compatible mechanisms with range larger than two. Appendix C illustrates how partial knitness, but not knitness, is satisfied for the domain of strict single-peaked preferences with three alternatives.

# 2 The model

Let $N = \{1, 2, ..., n\}$ be a finite set of *agents* with $n \geqslant 2$ and $A$ be a set of *alternatives.*

Let $\widetilde{\mathcal{R}}$ be the set of all complete, reflexive, and transitive binary relations on $A$ and $\mathcal{R}_i \subseteq \widetilde{\mathcal{R}}$ be the set of those preferences that are allowed for individual $i$. While $R_i \in \widetilde{\mathcal{R}}$ denotes agent $i$'s preferences, let $P_i$ and $I_i$ be the strict and the indifference part of $R_i$, respectively. For any $B \subseteq A$, $x \in B$, and $R_i \in \widetilde{\mathcal{R}}$, $U_B(R_i, x) = \{y \in B : yR_ix\}$ is the *(weak) upper contour set of* $R_i$ *at* $x$ *in* $B$ and $\overline{U}_B(R_i, x) = \{y \in B : yP_ix\}$ is the *strict upper contour set of* $R_i$ *at* $x$ *in* $B$.

Consider the following relationship between preferences.

**Definition 1** *We say that $R_i' \in \widetilde{\mathcal{R}}$ is an $x$-**monotonic transform in** $B$ **of** $R_i \in \widetilde{\mathcal{R}}$ if $U_B(R_i', x) \subseteq U_B(R_i, x)$ and $\overline{U}_B(R_i', x) \subseteq \overline{U}_B(R_i, x)$.*

In words, $R_i'$ is an $x$-monotonic transform in $B$ of $R_i$ if there exists a subset of $x$'s indifference class in $B$ of $R_i$ containing $x$, such that the relative position of its elements has weakly improved when going from $R_i$ to $R_i'$. A special class of monotonic transforms that are easy to identify are those where two preference relations have exactly the same weak and also the same strict upper contour sets for a given alternative $x$. Then we say that they are *reshufflings* of each other, and each of the two preferences are, in particular, monotonic transforms of the other.

Elements $R = (R_1, ..., R_n)$ in $\times_{i \in N} \mathcal{R}_i$ are called *preference profiles.*

Each agent $i \in N$ is endowed with a *type* $\theta_i$ belonging to a set $\Theta_i$. Each $\theta_i$ includes all the information in the hands of $i$. We denote by $\Theta = \times_{i \in N} \Theta_i$ the set of type profiles. A *type profile* is an $n$-tuple $\theta = (\theta_1, ..., \theta_n) \in \Theta$ that we will write as $\theta = (\theta_C, \theta_{N \setminus C})$ when we want to stress the role of coalition $C$ in $N$.

Once type profiles are fully determined, so are agents' preferences. We formalize this dependence through the notion of a *preference function.*

**Definition 2** *Let $\Theta$ be a set of type profiles. A* ***preference function*** *$\mathfrak{R}$ on $\Theta$, $\mathfrak{R} : \Theta \to \times_{i \in N} \mathcal{R}_i$, assigns a preferences profile $\mathfrak{R}(\theta)$ to each type profile $\theta \in \Theta$.*

We call $\mathfrak{R}(\theta) = (R_1(\theta), ..., R_n(\theta))$ the preference profile induced by the type profile $\theta$ while $R_i(\theta) \in \mathcal{R}_i$ stands for the induced preferences of agent $i$ at $\theta$. As usual $P_i(\theta)$ and $I_i(\theta)$ denote the strict and the indifference part of $R_i(\theta)$, respectively. Notice that $\mathcal{R}_i$ may be different for each agent (for example, in economies with private goods when individuals are selfish). Moreover, the domain of the preference function $\mathfrak{R}$ is a Cartesian product including all possible type profiles, but its range may be a non-Cartesian strict subset of $\times_{i \in N} \mathcal{R}_i$.

An *environment* is a pair $(\Theta, \mathfrak{R})$ formed by a set of type profiles and a preference function. Following standard use, *private values environments* are those where each agent's component of the preference function only depends on her type. That is, $R_i(\theta) = R_i(\theta_i, \theta'_{N \setminus \{i\}})$ for each agent $i \in N$, $\theta \in \Theta$, and $\theta'_{N \setminus \{i\}} \in \times_{j \in N \setminus \{i\}} \Theta_j$. Otherwise, we are in *interdependent values environments.* In private values environments, abusing notation, we will write $R_i(\theta_i)$ instead of $R_i(\theta)$.

Elements in the range of a preference function may be restricted to satisfy further conditions. In particular, an *environment* $(\Theta, \mathfrak{R})$ is *strict* if for any $\theta \in \Theta$ and any agent $i \in N$, $R_i(\theta) \in \mathcal{R}_i$ is a strict preference (that is, when the preferences of all agents under all type profiles are strict).

Our results refer to direct mechanisms. In fact, the properties we discuss are best analyzed with reference to the direct mechanism associated to any general one that might be described in terms of different message spaces and outcome functions.

A *direct mechanism* (*on* $\Theta$) is a function $f : \Theta \to A$. From now on, we drop the term "direct" and the reference to the set of type profiles and simply talk about mechanisms, without danger of ambiguity.

Notice that, by letting $\Theta$ be the domain of $f$, we implicitly assume that all type profiles within this set are considered to be feasible by the designer. We say that a mechanism $f$ has *full range* if the range of $f$ is $A$.

## 2.1 Properties of mechanisms

We now define properties of the mechanisms. We first look at incentives.

**Definition 3** *Let $(\Theta, \mathfrak{R})$ be an environment. We say that* ***an agent*** $i \in N$ ***can ex post profitably deviate under mechanism*** $f$ ***at*** $\theta \in \Theta$ *if there exists* $\theta'_i \in \Theta_i$ *such that* $f(\theta'_i, \theta_{N\setminus\{i\}})P_i(\theta)f(\theta)$. *A mechanism* $f$ *is* ***ex post incentive compatible*** *in* $(\Theta, \mathfrak{R})$ *if no agent can ex post profitably deviate at any type profile.*[3]

Therefore, the play where all agents reveal their true type must be a Nash equilibrium of the revelation game induced by the environment $(\Theta, \mathfrak{R})$.

In addition to individuals, coalitions of agents may also jointly deviate if they find it profitable. This leads us to propose the following definition.

**Definition 4** *Let $(\Theta, \mathfrak{R})$ be an environment. We say that* ***a coalition*** $C \subseteq N$ ***can ex post profitably deviate under mechanism*** $f$ ***at*** $\theta \in \Theta$ *if there exists* $\theta'_C \in \times_{i\in C}\Theta_i$ *such that for all agent* $i \in C$, $f(\theta'_C, \theta_{N\setminus C})R_i(\theta)f(\theta)$ *and for some* $j \in C$, $f(\theta'_C, \theta_{N\setminus C})P_j(\theta)f(\theta)$. *A mechanism* $f$ *is* ***ex post group incentive compatible*** *in* $(\Theta, \mathfrak{R})$ *if no coalition of agents can ex post profitably deviate at any type profile.*[4]

Notice that we allow for some agents to participate in the profitable deviation without strictly gaining from it and also for others to gain without changing their types. That facilitates the deviation by groups.

Remark that ex post individual and group incentive compatibility are purely ordinal. Since we concentrate on these properties, our whole framework is expressed in ordinal terms.

The next property is of a different nature. It essentially demands that for some specific changes in type profiles, no agent should affect the outcome (for her and for others) unless she changes her level of satisfaction.

**Definition 5** *Let $(\Theta, \mathfrak{R})$ be an environment. A mechanism* $f$ *is* ***(outcome) respectful*** *in* $(\Theta, \mathfrak{R})$ *if*

$$f(\theta)I_i(\theta)f(\theta'_i, \theta_{N\setminus\{i\}}) \text{ implies } f(\theta) = f(\theta'_i, \theta_{N\setminus\{i\}}),$$

*for each* $i \in N$, $\theta \in \Theta$, *and* $\theta'_i \in \Theta_i$ *such that* $R_i(\theta'_i, \theta_{N\setminus\{i\}})$ *is a* $f(\theta)$*-monotonic transform of* $R_i(\theta)$.[5]

For short, we call this condition respectfulness. Admittedly, this is a technical property, similar to those imposed in the literature when dealing with environments where agents' preferences allow for non-degenerate indifference classes (see Thomson, 2016). Relative to other technical conditions of the same sort, ours is among the weakest, because it only applies to some limited changes in type profiles. More importantly, it has no bite for strict environments which encompass some interesting cases (for example, in public good economies where agents' preferences are strict).

The following efficiency-related conditions will be used in our discussion of results.

[3] This property is called uniform incentive compatibility by Holmstrom and Myerson (1983). See also Bergemann and Morris (2005).

[4] Che, Kim, and Kojima (2015) consider group manipulations using the concept of first order stochastical dominance.

[5] Respectfulness is an analogous condition to the one we use in Barberà, Berga, and Moreno (2016) but requiring here invariance in outcomes instead of indifferences in outcomes. Examples of mechanisms satisfying respectfulness are provided in Section 6.

**Definition 6** *Let* $(\Theta, \mathfrak{R})$ *be an environment. A mechanism* $f$ *is* ***Pareto efficient on its range*** *in* $(\Theta, \mathfrak{R})$ *if for all* $\theta \in \Theta$*, there is no alternative* $x$ *in the range of* $f$ *such that* $xR_i(\theta)f(\theta)$ *for all* $i \in N$ *and* $xP_j(\theta)f(\theta)$ *for some* $j \in N$*.*

Notice that this is a relative notion of Pareto efficiency, whose significance is limited in the case where the range of a mechanism never contains certain outcomes that could eventually dominate those in a restricted range. Hence the need to single out those mechanisms that have full range and then satisfy the normatively attractive condition of Pareto efficiency.

**Definition 7** *Let* $(\Theta, \mathfrak{R})$ *be an environment. A mechanism* $f$ *is* ***Pareto efficient*** *in* $(\Theta, \mathfrak{R})$ *if the mechanism is Pareto efficient on its range in* $(\Theta, \mathfrak{R})$ *and* $f$ *has full range.*

Notice that ex post group incentive compatibility implies Pareto efficiency on the range, since otherwise the grand coalition could profitably deviate. But it does not immediately imply Pareto efficiency, unless accompanied by an argument or an assumption regarding the mechanism's range.

From now on we can omit reference to the environments on properties of $f$ when no confusion arises.

## 2.2 Conditions on environments

Until now, we have concentrated on the mechanisms. We now turn attention to introduce two conditions of environments that we call knit and partially knit and how they relate to each other. The consequences of defining mechanisms on environments that satisfy these conditions are discussed in Sections 3 and 4, while in Section 5 we discuss about their nature.

Both of them require a form of connectedness, given a preference function, between certain pairs of elements in the environment each one formed by a type profile and its induced preference profile. Specifically, for each of these pairs to be connected there must exist a third element that can be reached from each of the original two through "appropriate" sequences of elements, also in the environment.

Here are the sequences of types and preference profiles that are involved in the definition of our restrictions on environments.

All of their elements will be related to each other so that each one differs from its predecessor and its follower in the type of only one individual.

Let $S = \left\{\theta^S_{i(S,1)}, ..., \theta^S_{i(S,t_S)}\right\}$ be a sequence of individual types of length $t_S$, such that for each $h \in \{1, ..., t_S\}$, $\theta^S_{i(S,h)} \in \Theta_{i(S,h)}$. Agents may appear in that sequence several times or not at all. $I(S) = \{i(S, 1), ..., i(S, t_S)\}$ is the sequence of agents whose types appear in $S$ and $i(S, h)$ is the agent in position $h$ in $S$.

Given $\theta \in \Theta$ and $S = \left\{\theta^S_{i(S,1)}, ..., \theta^S_{i(S,t_S)}\right\}$, we consider the sequence of type profiles $m^h(\theta, S)$ that results from changing one at a time the types of agents according to $S$, starting from $\theta$. Formally, $m^h(\theta, S) \in \Theta$ is defined recursively so that $m^0(\theta, S) = \theta$ and for each $h \in \{1, ..., t_S\}$, $m^h(\theta, S) = \left(\left(m^{h-1}(\theta, S)\right)_{N \backslash i(S,h)}, \theta^S_{i(S,h)}\right)$.

Let $\theta \in \Theta$, and $S = \left\{\theta^S_{i(S,1)}, ..., \theta^S_{i(S,t_S)}\right\}$. We call the sequence of type profiles $\left\{m^h(\theta, S)\right\}_{h=0}^{t_S}$ **the passage from** $\theta$ **to** $\theta'$ **through** $S$ if $m^{t_S}(\theta, S) = \theta'$ for $\theta' \in \Theta$. More informally, we say that $\theta$ leads to $\theta'$ through $S$.

Notice that a given passage from $\theta$ to $\theta'$ through $S$ induces a corresponding sequence of preference profiles, $R^h(\theta, S) = (R^h_1(\theta, S), ..., R^h_n(\theta, S)) \in \times_{i \in N} \mathcal{R}_i$ for $h \in \{0, 1, ..., t_S\}$ where for each agent $i \in N$, we define $R^h_i(\theta, S) \equiv R_i\left(m^h(\theta, S)\right) \in \mathcal{R}_i$, that is, as the $i$th component of the preference function at the type profile $m^h(\theta, S)$.

We can now establish a condition (Definition 8) on the connection between sequences of changes in type profiles and the changes in preference profiles that they induce by means of the preference function.

Before that, remark that all along the paper, when we refer to Definitions 8 to 11 and make no reference to $B$, we implicitly assume that $B = A$. We only use the reference to a strict subset $B$ in $A$ in Subsection 3.1.

**Definition 8** *Let $B \subseteq A$, $x \in B$, and $\theta, \theta' \in \Theta$. We will say that the passage from $\theta$ to $\theta'$ through $S$ is $x$-**satisfactory in** $B$ if for each $h \in \{1, ..., t_S\}$, $R^h_{i(S,h)}(\theta, S)$ is an $x$-monotonic transform in $B$ of $R^{h-1}_{i(S,h)}(\theta, S)$.*

We say that $x$ is the reference alternative when going from $\theta$ to $\theta'$.[6]

Armed with our previous definitions we now say when two pairs, each of them formed by an alternative and a type profile, are *pairwise knit*. Whether they are or not will depend on how the preference function determines what sequences are satisfactory.

**Definition 9** *Let $B \subseteq A$ and $(\Theta, \mathfrak{R})$ be the environment. Two pairs formed by an alternative and a type profile each, $(x, \theta)$ and $(z, \widetilde{\theta})$, are **pairwise knit in** $B$ **and** $(\Theta, \mathfrak{R})$ if $(x, \theta)$, $(z, \widetilde{\theta}) \in B \times \Theta$, $\theta \neq \widetilde{\theta}$, and there exist $\theta' \in \Theta$ and sequences of types $S$ and $\widetilde{S}$, such that the passage from $\theta$ to $\theta'$ through $S$ is $x$-satisfactory in $B$ and the passage from $\widetilde{\theta}$ to $\theta'$ through $\widetilde{S}$ is $z$-satisfactory in $B$.*

We shall now define knit environments. This condition requires that any two pairs formed by an alternative and a type profile must be pairwise knit.

**Definition 10** *Let $B \subseteq A$. We say that an environment $(\Theta, \mathfrak{R})$ is **knit in** $B$ if any two pairs $(x, \theta)$, $(z, \widetilde{\theta}) \in B \times \Theta$, $\theta \neq \widetilde{\theta}$, $x \neq z$ are pairwise knit in $B$ and $(\Theta, \mathfrak{R})$.*

Our next definition refers to a condition on environments that is weaker than knitness, because not all pairs in them need to be pairwise knit. Given any profile $\theta$ and any alternative $x$, partial knitness only requires a comparison between $x$ and those alternatives $z$ that belong to its upper contour set of $x$ for two or more agents and are in the strict upper contour set for at least one agent. Specifically, the relevant pairs to be pairwise connected will be determined by the following two sets of agents: for any $\theta \in \Theta$ and $x, z \in B$, let $C(\theta, z, x) = \{i \in N : zR_i(\theta)x\}$ and $\overline{C}(\theta, z, x) = \{j \in N : zP_j(\theta)x\}$.

[6] The precise order of agents $I(S)$ may be crucial in the case of interdependent values. By contrast, in the case of private values the order of individuals in $S$ could be changed and the new sequence would still serve the same purpose.

**Definition 11** *Let $B \subseteq A$. We say that an environment $(\Theta, \mathfrak{R})$ is* ***partially knit in*** *$B$ if any two pairs formed by an alternative and a type profile each, $(x,\theta)$, $(z,\widetilde{\theta}) \in A \times \Theta$, $\theta \neq \widetilde{\theta}$, such that $\overline{C}(\theta,z,x) \neq \emptyset$, $\#C(\theta,z,x) \geq 2$, and $\widetilde{\theta}_j = \theta_j$ for any $j \in N \backslash C(\theta,z,x)$ are pairwise knit in $B$ in $(\Theta,\mathfrak{R})$.*

We end this subsection with Example 1 which is an adaptation of Example 1 in Bergemann and Morris (2005). It illustrates how the distinction between satisfactory and non-satisfactory passages arises in an interdependent values environment and has an impact on whether or not an environment is knit.

**Example 1** *Let $N = \{1,2\}$ and $A = \{a,b,c\}$. Each agent $i$ has two possible types: $\Theta_i = \{\underline{\theta}_i, \overline{\theta}_i\}$. The preference function $\mathfrak{R}$ is defined in Table 1. We write, in each cell, the preferences of both agents for a given type profile represented by an ordered list from better to worse, with parenthesis in case of indifferences. Observe that agent 2's preferences over b and c depend on agent 1's type: $bP_2(\underline{\theta}_1,\underline{\theta}_2)c$ while $cP_2(\overline{\theta}_1,\underline{\theta}_2)b$, that is, we are in an interdependent values environment.*

| $\mathfrak{R}$ | $\underline{\theta}_2$ | | $\overline{\theta}_2$ | |
|---|---|---|---|---|
| $\underline{\theta}_1$ | $R_1(\underline{\theta}_1,\underline{\theta}_2)$ *acb* | $R_2(\underline{\theta}_1,\underline{\theta}_2)$ *b(ac)* | $R_1(\underline{\theta}_1,\overline{\theta}_2)$ *bca* | $R_2(\underline{\theta}_1,\overline{\theta}_2)$ *a(bc)* |
| $\overline{\theta}_1$ | $R_1(\overline{\theta}_1,\underline{\theta}_2)$ *c(ab)* | $R_2(\overline{\theta}_1,\underline{\theta}_2)$ *c(ab)* | $R_1(\overline{\theta}_1,\overline{\theta}_2)$ *c(ab)* | $R_2(\overline{\theta}_1,\overline{\theta}_2)$ *c(ab)* |

*Table 1. Preference function for Example 1.*

*Notice that the range of $\mathfrak{R}$ is not a Cartesian product, since $\mathcal{R}_1 = \{acb, bca, c(ab)\}$ and $\mathcal{R}_2 = \{b(ac)), a(bc), c(ab)\}$ but the preferences profile $(acb, a(bc))$ is not in the range of the preference function $\mathfrak{R}$.*
*Let $x = a$, $\theta = (\underline{\theta}_1,\underline{\theta}_2)$, $\theta' = (\overline{\theta}_1,\underline{\theta}_2)$, and $S = \left\{\overline{\theta}_2,\overline{\theta}_1,\underline{\theta}_2\right\}$ be a sequence of individual types. Note that, $I(S) = \{2,1,2\}$ and $t_S = 3$. The passage from $\theta$ to $\theta'$ through $S$ is a-satisfactory as proven in Remark 4 in Appendix B. Let $x = a$, $\theta = (\underline{\theta}_1,\underline{\theta}_2)$, $\theta' = (\overline{\theta}_1,\overline{\theta}_2)$, and $S = \left\{\overline{\theta}_1,\overline{\theta}_2\right\}$ be a sequence of individual types. Note that, $I(S) = \{1,2\}$ and $t_S = 2$. The passage from $\theta$ to $\theta'$ through $S$ is not a-satisfactory as proven in Remark 4 in Appendix B.*
*In Remark 5 in Appendix B we show that the environment in this example is knit.*

# 3 An impossibility result

We now present our first result and discuss its consequences. The result in Theorem 1 states that if the environment is knit only constant mechanisms can be ex post incentive compatible and respectful.

**Theorem 1** *Let $(\Theta,\mathfrak{R})$ be any knit environment and $f : \Theta \to A$ be any mechanism. If $f$ is ex post incentive compatible and respectful, then $f$ is constant.*

**Proof of Theorem 1.** Let $(\Theta, \mathfrak{R})$ be a knit environment and let $f$ be an ex post incentive compatible and respectful mechanism. Assume, by contradiction, that $f$ is not constant. Then, there will be $x, z \in A$, $x \neq z$ such that $x = f(\theta)$ and $z = f(\widetilde{\theta})$ for some $\theta$ and $\widetilde{\theta}$ in $\Theta$. Since $(\Theta, \mathfrak{R})$ is knit, the two pairs formed by an alternative and a type profile, $(x, \theta)$ and $(z, \widetilde{\theta}) \in A \times \Theta$, are pairwise knit. Thus, there exist $\theta' \in \Theta$ and two sequences $S = \{\theta^S_{i(S,1)}, ..., \theta^S_{i(S,t_S)}\}$, $\widetilde{S} = \{\widetilde{\theta}^{\widetilde{S}}_{i(\widetilde{S},1)}, ..., \widetilde{\theta}^{\widetilde{S}}_{i(\widetilde{S},t_{\widetilde{S}})}\}$ such that the passage from $\theta$ to $\theta'$ through $S$ is $x$-satisfactory and the passage from $\widetilde{\theta}$ to $\theta'$ through $\widetilde{S}$ is $z$-satisfactory.
Now, we will show the following:
(a) for each $h \in \{1, ..., t_S\}$, $f(m^h(\theta, S)) = x$, and
(b) for each $h \in \{1, ..., t_{\widetilde{S}}\}$, $f(m^h(\widetilde{\theta}, \widetilde{S})) = z$.
Statements in (a) and (b) yield to a contradiction. By definition of the sequences $S$ and $\widetilde{S}$, we know that $m^{t_S}(\theta, S) = m^{t_{\widetilde{S}}}(\widetilde{\theta}, \widetilde{S}) = \theta'$. However, $f(\theta') = f(m^{t_S}(\theta, S)) = x$ by (a) while $f(\theta') = f(m^{t_{\widetilde{S}}}(\widetilde{\theta}, \widetilde{S})) = z$ by (b).
We prove (a) in steps, from $h = 1$ to $h = t_S$. The proof of (b) is identical and omitted.
Step 1. Let $h = 1$. By Definition 8, $R^1_{i(S,1)}(\theta, S)$ is an $x$-monotonic transform of $R^0_{i(S,1)}(\theta, S) = R_{i(S,1)}(\theta)$. (1)
Observe that $f(m^1(\theta, S)) \in U\left(R^1_{i(S,1)}(\theta, S), x\right)$. (2)
(otherwise, if $f(m^1(\theta, S)) \notin U\left(R^1_{i(S,1)}(\theta, S), x\right)$, we would get a contradiction to ex post incentive compatibility since $i(S, 1)$ would ex post profitably deviate under $f$ at $(\theta^S_{i(S,1)}, (m^0(\theta, S))_{N\setminus\{i(S,1)\}})$ via $\theta_{i(S,1)}$).
By (1) and (2) we have that $f(m^1(\theta, S)) \in U\left(R^0_{i(S,1)}(\theta, S), x\right)$. (3)
By ex post incentive compatibility of $f$, $f(m^1(\theta, S)) \notin \overline{U}\left(R^0_{i(S,1)}(\theta, S), x\right)$. (4)
(otherwise, if $f(m^1(\theta, S)) \in \overline{U}\left(R^0_{i(S,1)}(\theta, S), x\right)$, $f(m^1(\theta, S))P^0_{i(S,1)}(\theta)x$ contradicting ex post incentive compatibility since $i(S, 1)$ would ex post profitably deviate under $f$ at $\theta$ via $\theta^S_{i(S,1)}$).
Thus, by (3) and (4) we have that $f(m^1(\theta, S))$ is indifferent to $x$ according to preference $R^0_{i(S,1)}(\theta, S)$ (that is, $f(m^1(\theta, S))I^0_{i(S,1)}(\theta, S)x$). (5)
Then, by respectfulness, we get that $f(m^1(\theta, S)) = f(m^0(\theta, S)) = f(\theta) = x$ which ends the proof of (a) for $h = 1$.
Step $h \in \{2, ..., t_S\}$. By repeating the same argument than in Step 1 on the recursive fact that $f(m^{h-1}(\theta, S)) = x$, we obtain that $f(m^h(\theta, S)) = f(m^{h-1}(\theta, S)) = x$. ■

Since respectfulness does not have bite for strict environments the following Corollary 1 results straightforwardly by applying the first part in each step of the proof of Theorem 1 where respectfulness is not used.

**Corollary 1** *Let $(\Theta, \mathfrak{R})$ be any strict knit environment and $f : \Theta \to A$ be any mechanism. If $f$ is ex post incentive compatible, then $f$ is constant.*

Let us discuss the content of Theorem 1 and how it fits in the literature. A first observation is that knitness is a sufficient condition for the result to hold. In Subsection 3.1 below we present the partial results we have about the necessity of knitness for the impossibility result to hold.

Our impossibility result extends and unifies several ones proving that only constant functions can be ex post incentive compatible under different conditions, in the case of interdependent values. This first and very negative result is strongly reminiscent of previous ones obtained in different contexts. The conclusion of Theorem 1 is in the same vein than the one that Jehiel, Meyer-Ter-Vehn, Moldovanu, and Zame (2006) obtain under completely different premises. These authors focus on environments where preference profiles will be represented by $n$-tuples of money-separable utility functions, and where the preference function is smooth, among other assumptions, while our restrictions apply to environments that do not have such characteristics. Other papers arrive at the same conclusion than Theorem 1 in the context of more specific models. These include the models considered by Che, Kim, and Kojima (2015) for a house allocation problem[7] without transfers (see their Theorem 1), Austen-Smith and Feddersen (2006) for the voting in the deliberative jury problem (see their Theorem and Corollary), Dasgupta and Maskin (2000) in auctions (see their Example 4), and see also Examples 2 and 4 in our Subsection 6.2 below. Each of the previous papers that came to the same conclusion used assumptions that were specific to a given application, including those on the domain of definition of the mechanisms under scrutiny. Our condition of knitness is based in a careful analysis of the reasoning underlying the different and hard to compare papers that arrived that common conclusion. As a result, it may lose the flavor that its counterparts did have in each specific application, since it emerges from a theoretical effort to distill the essential aspects of each one without the pretense of bringing all of them together. But, as a proof of its significance, we also submit another partial but important consequence of imposing knitness as a property of environments. We prove that is not only a sufficient condition to precipitate the negative results that we have found scattered in the literature, but also a necessary one, when only two alternatives are in the range of a mechanism. Hence, one can see that it is not only a theoretical construct, but definitely an exact identification of what it takes to obtain the result at hand in this specific but important case. This statement will be proved in the next section as Theorem 2.

Since the results and the proofs in Che, Kim, and Kojima (2015) hold strong similarities with ours, when applied to their specific context it is worthwhile to point at their parallels and their differences. A major similarity is that both, their paper and ours, use of a technique that was introduced by Gibbard when proving his celebrated impossibility theorem which consists in passing from one profile of characteristics to another by successive changes in those of one agent at a time. Assumption 3 in Che, Kim, and Kojima (2015) allows these authors to select appropriately connected sequences of that sort. Yet notice that, since a change in the type of a single agent can induce changes in the preferences of others, the sequences of preference profiles that are naturally associated to the sequences of types may involve preference changes in more than one agent at each step. Hence, their environments satisfy our notion of knitness in a specific form, since our definition of knitness does not require the pair of type profiles to be directly connected. On the other hand, when changing the type of one single agent we do not impose invariance of this agent's preferences as Che,

[7] A house allocation problem considers a set of individuals that must be assigned a maximum of one house each, out of a set of houses.

Kim, and Kojima (2015) do: we admit that they may change in a monotonic manner. In spite of this analogy in the technique of proof, and of the fact that our formal setting refers to a more general and abstract family of environments, our paper differs from Che, Kim, and Kojima (2015)'s in that they use different assumptions. Their Assumptions 2 and 4 require a richness of type profiles, that we do not demand, which yields richness in preferences profiles. In addition, their Theorems 1 and 2 invoke the Pareto condition: in Theorem 1 the requirement is explicit, while in Theorem 2 it is a combined consequence of ex post group incentive compatibility and the richness domain condition in Assumption 2. All these conditions allow to reduce the set of environments to consider to be considered to those where as two agents-two objects' problem is to be solved. In this reduced problem, their Assumption 3 implies that such environments are knit (see Example 4 for an application). Also, in this reduced problem, a weaker version of respectfulness holds, although it is not imposed, where the monotonic transform is that agent's ordinal preferences are the same and strict for the agent changing her type. Thus, the only way an agent is indifferent between two alternatives is when she gets the same, and therefore, the other agent too.

Two final remarks are in order. Observe that there is no contradiction between our result in Theorem 1 that only constant mechanisms are strategy-proof and that of the Gibbard-Satterthwaite theorem which also admits dictatorship, since the universal set of preferences where the latter applies is not knit, as shown in Proposition 1, and thus Theorem 1 does not apply.

Notice also that, since we work with single valued direct mechanisms, our environments are separable in the sense of Bergemann and Morris (2005), and their Corollary 1 applies: no mechanism is interim incentive compatible unless it is ex post incentive compatible. Because of that, Theorem 1 and Corollary 1 have direct implications on the weaker interim notion, with no need to be explicit about agents' beliefs.

## 3.1 The tightness of the impossibility result

Is it possible to relax the knitness requirement and still obtain the conclusion that ex post incentive compatible and respectful mechanisms must be constant? If we restrict ourselves to binary mechanisms, then we can provide a definite and negative answer to that question. In that case, knitness in any two-elements' set, is a necessary and sufficient condition for the constancy of ex post incentive compatible and respectful mechanisms and therefore our Theorem 1 is tight.

Theorem 2 states that for any environment that is not knit in some pair of alternatives, one can find binary and thus not constant mechanisms with this limited range and satisfy the rest of properties.

**Theorem 2** *An environment* $(\Theta, \mathfrak{R})$ *admits a binary ex post incentive compatible and respectful mechanism with range* $B = \{x, z\} \subseteq A$ *if and only if it is not knit in* $B$.

Before proving Theorem 2, it is convenient to remark the relationship between knitness (in $A$) and knitness in $B$, for any $B \subsetneq A$.

**Remark 1** *If an environment $(\Theta, \mathfrak{R})$ is knit then it is knit in $B$ for all $B \subsetneq A$. The proof is straightforward by definition, by the fact that an $x$-monotonic transform in $A$ is also an $x$-monotonic transform in $B$ for any $B \subsetneq A$, $x \in B$. The converse result does not hold as Example 7, Appendix B shows: the second environment that we define is knit for any two-elements set $B \subsetneq A$ but it is not knit. In this second environment, there is no binary ex post incentive compatible mechanism. In the first environment in Example 7 there exist binary ex post incentive compatible mechanism. As shown in the following theorem, the reason is that the first environment is not knit for some two-elements set $B \subsetneq A$.*

In the light of this remark, in Theorem 1 we only impose knitness in $A$ because it implies knitness in $B$ for any $B \subsetneq A$.
**Proof of Theorem 2.** Let us prove, by construction, that if an environment $(\Theta, \mathfrak{R})$ is not knit in $B = \{x, z\} \subseteq A$, then there exist a binary ex post incentive compatible and respectful mechanism on $(\Theta, \mathfrak{R})$ with range $B$. By not knitness in $B$, there are two pairs formed by an alternative and a type profile each, $(x, \theta)$, $(z, \widetilde{\theta}) \in B \times \Theta$, $\theta \neq \widetilde{\theta}$, $x \neq z$ such that there do not exist $\theta' \in \Theta$, $S$, $\widetilde{S}$ where the passage from $\theta$ to $\theta'$ through $S$ is $x$-satisfactory in $B$ and the passage from $\widetilde{\theta}$ to $\theta'$ through $\widetilde{S}$ is $z$-satisfactory in $B$.

Before defining the desired mechanism, let us first propose the following partition of $\Theta$: $\Theta_1 = \{\overline{\theta} \in \Theta :$ there is $S$ such that the passage from $\theta$ to $\overline{\theta}$ through $S$ is $x$-satisfactory in $B\}$, $\Theta_2 = \{\overline{\theta} \in \Theta :$ there is $\widetilde{S}$ such that the passage from $\widetilde{\theta}$ to $\overline{\theta}$ through $\widetilde{S}$ is $z$-satisfactory in $B\}$, $\Theta_3 = \Theta \backslash (\Theta_1 \cup \Theta_2)$. Note that since the environment is not knit in $B$, $\Theta_1 \cap \Theta_2 = \varnothing$.[8]
We now define a mechanism $f$ as follows: $f(\widehat{\theta}) = x$ if $\widehat{\theta} \in \Theta_1 \cup \Theta_3$ and $f(\widehat{\theta}) = z$, otherwise. Let us first check, by contradiction, that $f$ is ex post incentive compatible.
Suppose that there exist $\overline{\theta} \in \Theta$, agent $i \in N$, $\theta_i' \in \Theta$ such that $f(\theta_i', \overline{\theta}_{N\backslash\{i\}}) P_i(\overline{\theta}) f(\overline{\theta})$.
<u>Case 1</u>. $f(\overline{\theta}) = x$ and $f(\theta_i', \overline{\theta}_{N\backslash\{i\}}) = z$. Thus, by definition of $f$, either (1) $\overline{\theta} \in \Theta_1$ or (2) $\overline{\theta} \in \Theta \backslash (\Theta_1 \cup \Theta_2)$. However, $(\theta_i', \overline{\theta}_{N\backslash\{i\}}) \in \Theta_2$.
Observe that $R_i(\overline{\theta})$ is such that $z P_i(\overline{\theta}) x$ and $R_i(\theta_i', \overline{\theta}_{N\backslash\{i\}})$ can be any preference. That is, either $z R_i(\theta_i', \overline{\theta}_{N\backslash\{i\}}) x$, $x R_i(\theta_i', \overline{\theta}_{N\backslash\{i\}}) z$, or $x I_i(\theta_i', \overline{\theta}_{N\backslash\{i\}}) z$ holds. For the three cases, observe that the passage from $\overline{\theta}$ to $(\theta_i', \overline{\theta}_{N\backslash\{i\}})$ through $S' = \{\theta_i'\}$ where $I(S') = (i)$ is $x$-satisfactory in $B$. Also, the passage from $(\theta_i', \overline{\theta}_{N\backslash\{i\}})$ to $\overline{\theta}$ through $\widehat{S}$ where $I(\widehat{S}) = (i)$ is $z$-satisfactory in $B$.
Since $(\theta_i', \overline{\theta}_{N\backslash\{i\}}) \in \Theta_2$, we have that the passage from $\widetilde{\theta}$ to $(\theta_i', \overline{\theta}_{N\backslash\{i\}})$ through $\widetilde{S}$ is $z$-satisfactory in $B$ and therefore the passage from $\widetilde{\theta}$ to $\overline{\theta}$ through $\widetilde{S} \cup \widehat{S}$ is $z$-satisfactory in $B$. Therefore $\overline{\theta} \in \Theta_2$ which is a contradiction to the assumption that $\overline{\theta} \in \Theta_1 \cup \Theta_3$.
<u>Case 2</u>. $f(\theta_i', \overline{\theta}_{N\backslash\{i\}}) = x$ and $f(\overline{\theta}) = z$. Thus, by definition of $f$, either (1) $(\theta_i', \overline{\theta}_{N\backslash\{i\}}) \in \Theta_1$ or (2) $(\theta_i', \overline{\theta}_{N\backslash\{i\}}) \in \Theta \backslash (\Theta_1 \cup \Theta_2)$. However, $\overline{\theta} \in \Theta_2$.
Observe that $R_i(\overline{\theta})$ is such that $x P_i(\overline{\theta}) z$ and $R_i(\theta_i', \overline{\theta}_{N\backslash\{i\}})$ can be any preference. That is, either $z R_i(\theta_i', \overline{\theta}_{N\backslash\{i\}}) x$, $x R_i(\theta_i', \overline{\theta}_{N\backslash\{i\}}) z$, or $x I_i(\theta_i', \overline{\theta}_{N\backslash\{i\}}) z$ holds. For the three cases, observe that the passage from $\overline{\theta}$ to $(\theta_i', \overline{\theta}_{N\backslash\{i\}})$ through $S' = \{\theta_i'\}$ where $I(S') = (i)$ is $z$-satisfactory in $B$. Also, the passage from $(\theta_i', \overline{\theta}_{N\backslash\{i\}})$ to $\overline{\theta}$ through $\widehat{S}$ where $I(\widehat{S}) = (i)$ is a $x$-satisfactory in $B$.

[8] The partition of $\Theta$ is simpler but it coincides with the one used in Barberà, Berga, and Moreno (2019).

Since $\overline{\theta} \in \Theta_2$, we have that the passage from $\widetilde{\theta}$ to $\overline{\theta}$ through $\widetilde{S}$ is $z$-satisfactory in $B$ and therefore the passage from $\widetilde{\theta}$ to $(\theta'_i, \overline{\theta}_{N\setminus\{i\}})$ through $\widetilde{S} \cup S'$ is $z$-satisfactory in $B$. Therefore, $(\theta'_i, \overline{\theta}_{N\setminus\{i\}}) \in \Theta_2$ which is a contradiction to the assumption that $(\theta'_i, \overline{\theta}_{N\setminus\{i\}}) \in \Theta_1 \cup \Theta_3$.

Now, we show that $f$ is respectful.

By contradiction suppose that there exist $\overline{\theta} \in \Theta$, agent $i \in N$, $\theta'_i \in \Theta$ such that $f(\theta'_i, \overline{\theta}_{N\setminus\{i\}}) I_i(\overline{\theta}) f(\overline{\theta})$, $f(\theta'_i, \overline{\theta}_{N\setminus\{i\}}) \neq f(\overline{\theta})$, and $R_i(\theta'_i, \overline{\theta}_{N\setminus\{i\}})$ is a $f(\overline{\theta})$-monotonic transform of $R_i(\overline{\theta})$.

First, assume that $f(\overline{\theta}) = a$ and $f(\theta'_i, \overline{\theta}_{N\setminus\{i\}}) = b$. Thus, by definition of $f$, either (1) $\overline{\theta} \in \Theta_1$ or (2) $\overline{\theta} \in \Theta\setminus(\Theta_1 \cup \Theta_2)$. However, $(\theta'_i, \overline{\theta}_{N\setminus\{i\}}) \in \Theta_2$.

Observe that $R_i(\overline{\theta})$ is such that $b I_i(\overline{\theta}) a$ and $R_i(\theta'_i, \overline{\theta}_{N\setminus\{i\}})$ can be any preference. That is, either $b R_i(\theta'_i, \overline{\theta}_{N\setminus\{i\}}) a$, $a R_i(\theta'_i, \overline{\theta}_{N\setminus\{i\}}) b$, or $a I_i(\theta'_i, \overline{\theta}_{N\setminus\{i\}}) b$ holds. For the three cases, observe that the passage from $\overline{\theta}$ to $(\theta'_i, \overline{\theta}_{N\setminus\{i\}})$ through $S' = \{\theta'_i\}$ where $I(S') = (i)$ is $a$-satisfactory. Also, the passage from $(\theta'_i, \overline{\theta}_{N\setminus\{i\}})$ to $\overline{\theta}$ through $\widehat{S}$ where $I(\widehat{S}) = (i)$ is a $b$-satisfactory.

Repeating the same argument as in Cases 1 and 2 above we get the desired contradiction.

An identical argument as above holds if $f(\theta'_i, \overline{\theta}_{N\setminus\{i\}}) = a$ and $f(\overline{\theta}) = b$.

To prove the second part of our result, let $f$ be a binary ex post incentive compatible and respectful mechanism with range $B = \{x, z\}$ and suppose that the environment $(\Theta, \mathfrak{R})$ is knit in $B = \{x, z\}$. We will obtain a contradiction.

By definition of $f$, $x = f(\theta)$ and $z = f(\widetilde{\theta})$ for some $\theta$ and $\widetilde{\theta}$ in $\Theta$. Since $(\Theta, \mathfrak{R})$ is knit in $B$, the two pairs formed by an alternative and a type profile, $(x, \theta)$ and $(z, \widetilde{\theta}) \in B \times \Theta$, are pairwise knit in $B$. Thus, there exist $\theta' \in \Theta$ and two sequences $S = \{\theta^S_{i(S,1)}, ..., \theta^S_{i(S,t_S)}\}$, $\widetilde{S} = \{\widetilde{\theta}^{\widetilde{S}}_{i(\widetilde{S},1)}, ..., \widetilde{\theta}^{\widetilde{S}}_{i(\widetilde{S},t_{\widetilde{S}})}\}$ such that the passage from $\theta$ to $\theta'$ through $S$ is $x$-satisfactory in $B$ and the passage from $\widetilde{\theta}$ to $\theta'$ through $\widetilde{S}$ is $z$-satisfactory in $B$.

Although these sequences are not necessarily the same than the ones we used in the proof of Theorem 1, from this point on, we can use the same reasoning as there, and show that

(a) for each $h \in \{1, ..., t_S\}$, $f(m^h(\theta, S)) = x$, and

(b) for each $h \in \{1, ..., t_{\widetilde{S}}\}$, $f(m^h(\widetilde{\theta}, \widetilde{S})) = z$,

leading to a contradiction. Two comments are relevant. First, that in each step $h$ the outcome is either $x$ or $z$. Second, we use the arguments we have already used in the proof of Theorem 1 taking into account that we must replace the (strict) upper set by the (strict) upper set in $B$, respectively. This would complete the proof for the present theorem. ■

Theorem 2 extends the result in Barberà, Berga, and Moreno (2019) to the case of binary mechanisms and any set of alternatives and offers a simpler proof of it. This result is important on its own, since there are important problems where it is enough to choose among two alternatives and already leads us to several conclusions. But it also supports the idea that the requirement of knitness is significant to obtain the negative results whose deep causes we are trying to understand. Clearly it is in the case of binary choice mechanisms, since then it is not only sufficient but also necessary. This necessity does not hold in general, when considering mechanisms with a larger range. But even then we can prove that a sufficiently large number of violations of knitness are needed for the existence of ex post incentive compatible and respectful mechanisms of a given range size. This number is a function of the size of the mechanism's range and the specific alternatives it contains, as stated by Theorem 3.

**Theorem 3** *An environment* $(\Theta, \Re)$ *admits an ex post incentive compatible and respectful mechanism with range* $B = \{x^1, ..., x^k\}$ *where* $x^l \neq x^m$ *for all* $l, m \in \{1, ..., k\}$*, only if there exist* $k$ *pairs* $(\theta^1, x^1)$*,* $(\theta^2, x^2)$*, ...,* $(\theta^k, x^k)$ *such that for all* $l, m \in \{1, ..., k\}$*,* $(\theta^l, x^l)$ *and* $(\theta^k, x^k)$ *are not pairwise knit in* $B$*.*

**Proof of Theorem 3.** Take an ex post incentive compatible and respectful mechanism of range $k$, $k$ higher than 2. Then, there exist at least one group of $k$ pairs $(\theta^1, x^1)$, $(\theta^2, x^2)$,..., $(\theta^k, x^k)$ where the mechanism associates $x^t$ to $\theta^t$ for each $t \in \{1, ...k\}$. Take any group of such $k$ pairs. We now show that any two pairs in this group are not pairwise knit. Suppose, by contradiction that for some $l, m \in \{1, ..., k\}$, $(\theta^l, x^l)$, $(\theta^m, x^m)$ are pairwise knit. Proceeding exactly as in the proof of Theorem 1 we get a contradiction to the fact that the mechanism is ex post incentive compatible and respectful. Repeating the same argument for any group of such $k$ pairs we obtain the desired conclusion. ■

Example 7 in Appendix B proves that the conditions of Theorem 3 are necessary but not sufficient. The example provides an environment satisfying them that may sometimes not allow escaping the constancy conclusion, while allowing it in other cases.

Hence, we believe that the study of knit environments is justified as part of a search for conditions that may allow for satisfactory ex post incentive compatible mechanisms. Yet, we must admit that although violation of knitness may open the door to more flexible binary mechanisms, this does not mean that all mechanisms that may emerge need to be attractive. In the first environment defined in Example 7 (Appendix B) knitness in B is violated only for B being a particular pair of alternatives, and the unique ex post incentive compatible mechanism is a dictator. But our Example 3 in Subsection 6.2.1, inspired in our reading of the work by Austen-Smith and Feddersen (2006), suggests that some specific non-knit environments may allow for the existence of attractive voting by quota mechanisms.

# 4 Equivalence between ex post individual and group incentive compatibility

Since the 1980's the literature on interdependent value environments has obtained positive and negative results regarding the possibility of designing ex post incentive compatible mechanisms. Most papers are mainly motivated by or applied to auctions. A number of authors have shown that ex post incentive compatibility and efficiency are compatible when signals are one-dimensional and a single-crossing property is satisfied: see Crémer and McLean (1985), Maskin (1992), Ausubel (1997), Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), Bergemann and Välimäki (2002), Perry and Reny (2002), for example. When it comes to situations where types are multi-dimensional, the general wisdom is that negative results prevail. Under additional assumptions, no mechanism or only constant ones are implementable: see for example Maskin (1992), Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), and Jehiel, Meyer-Ter-Vehn, Moldovanu, and Zame (2006). Dizdar and Moldovanu (2016) study a two-sided matching model with a finite number of agents, two-sided incomplete information, interdependent values, and multi-dimensional attributes and show that premuneration values corresponding to uniform, fixed-proportion sharing are essentially the only incentive compatible and efficient ones in their setting.

All of these papers refer to the individual notion of ex post incentive compatibility. With the notable exception of Che, Kim, and Kojima (2015), who consider group manipulations using first-order stochastic dominance, most of the literature stops short of analyzing the consequences of allowing for strategic deviations by groups. In this paper we introduce and study the more demanding condition of ex post group incentive compatibility, and show that respectful and ex post incentive compatible mechanisms defined on partially knit environments satisfy the stronger version of incentive compatibility for groups, as well. We discuss the consequences of this equivalence after presenting this result formally.

**Theorem 4** *Let* $(\Theta, \mathfrak{R})$ *be any partially knit environment and* $f$ *be any respectful mechanism in* $(\Theta, \mathfrak{R})$. *Then,* $f$ *is ex post incentive compatible in* $(\Theta, \mathfrak{R})$ *if and only if* $f$ *is ex post group incentive compatible in* $(\Theta, \mathfrak{R})$.

Part of the proof of Theorem 4 follows an identical reasoning used in the proof of Theorem 1. We write down the first part of the proof, which is the one that differs, and specify from where on the argument is the same.

**Proof of Theorem 4.** Let $(\Theta, \mathfrak{R})$ be a partially knit environment and let $f$ be a respectful mechanism. By definition, ex post group incentive compatibility implies ex post incentive compatibility. To prove the converse, suppose, by contradiction, that there exist $\theta \in \Theta$, $C \subseteq N$, $\#C \geq 2$, $\widetilde{\theta}_C \in \times_{i \in C} \Theta_i$ such that for any agent $i \in C$, $f(\widetilde{\theta}_C, \theta_{N \setminus C}) R_i(\theta) f(\theta)$ and $f(\widetilde{\theta}_C, \theta_{N \setminus C}) P_j(\theta) f(\theta)$ for some agent $j \in C$. Let $z = f(\widetilde{\theta}_C, \theta_{N \setminus C})$ and $x = f(\theta)$. Note that $(i)$ $z \neq x$, $(ii)$ $\overline{C}(\theta, z, x) \neq \varnothing$, $\#C(\theta, z, x) \geq 2$ since $C \subseteq C(\theta, z, x)$, and $(iii)$ $\widetilde{\theta}_j = \theta_j$ for any $j \in N \setminus C(\theta, z, x)$ again since $C \subseteq C(\theta, z, x)$.
Since $(\Theta, R)$ is partially knit and conditions in Definition 11 are satisfied, $(x, \theta)$ and $(z, \widetilde{\theta})$ are pairwise knit. Thus, there exist $\theta' \in \Theta$ and two sequences of types $S = \{\theta^S_{i(S,1)}, ..., \theta^S_{i(S,t_S)}\}$, $\widetilde{S} = \{\widetilde{\theta}^{\widetilde{S}}_{i(\widetilde{S},1)}, ..., \widetilde{\theta}^{\widetilde{S}}_{i(\widetilde{S},t_{\widetilde{S}})}\}$ such that the passage from $\theta$ to $\theta'$ through $S$ is $x$-satisfactory and the passage from $\widetilde{\theta}$ to $\theta'$ through $\widetilde{S}$ is $z$-satisfactory.
Although these sequences are not necessarily the same than the ones we used in the proof of Theorem 1, from this point on, we can use the same reasoning as there, and show that
(a) for each $h \in \{1, ..., t_S\}$, $f(m^h(\theta, S)) = x$, and
(b) for each $h \in \{1, ..., t_{\widetilde{S}}\}$, $f(m^h(\widetilde{\theta}, \widetilde{S})) = z$,
again leading to a contradiction. Adding the arguments we have already used in the proof of Theorem 1 we would complete the one for the present theorem. ■

Theorem 4 restricts attention to mechanisms that are respectful, but the latter requirement does not have bite when the environment is strict. Corollary 2 holds straightforwardly.

**Corollary 2** *Let* $(\Theta, \mathfrak{R})$ *be any partially knit strict environment and* $f$ *be any mechanism. Then,* $f$ *is ex post incentive compatible in* $(\Theta, \mathfrak{R})$ *if and only if* $f$ *is ex post group incentive compatible in* $(\Theta, \mathfrak{R})$.

Let us discuss the content and implications of Theorem 4. Partial knitness is only sufficient but not necessary for all respectful and ex post incentive compatible mechanisms to also be ex post group incentive compatible (see Example 6 in Appendix B). Corollary 2 reaches the same conclusion for strict environments without need to invoke respectfulness. The study of necessity is left for further research.

Theorem 4 applies to the case of interdependent and private values. This equivalence result when defined in partially knit environments is a result in the same line of those that we had proven only for the case of private values in Barberà, Berga, and Moreno (2010, 2016), showing the often ignored connection between individual and group strategic considerations. Notice that the equivalence will hold for all mechanisms defined on such environments, whether or not they are satisfactory from other points of view like Pareto efficiency. Therefore, more ambitious conclusions regarding the possibility to define satisfactory mechanisms within this class needs further analysis, and examples of partially knit environments satisfying further properties. Our search across the literature did not produce as many positive results for genuinely interdependent values contexts as it did for the case of private values (to which Theorem 4 also apply) but the examples we found come from different fields and in a variety of models. Some definitely exist in interdependent values, like Pourpouneh, Ramezanianz, and Sen, (2020)[9], along with Examples 3 and 5 in Section 5 below (inspired by Austen-Smith and Feddersen, 2006 and Che, Kim, and Kojima, 2015, respectively), and also examples in the particular case of private values like the ones studied in Propositions 2, 3, 4.

A first implication of ex post group incentive compatibility is Pareto efficiency on the mechanism's range. Admittedly, this is a limited result since it depends on the set of alternatives selected by the mechanism. Hence, the implications that having a good performance regarding incentives may be compatible with Pareto efficiency is an invitation to investigate those cases where this may be a promising possibility. The compatibility between incentives and Pareto efficiency does not always hold: see Theorem 1 in Che, Kim, and Kojima (2015), and for the case of ordinal preferences see Yamashita and Zhu (2021), among others. In the literature there are different papers where non-trivial mechanisms that are ex post incentive compatible and Pareto efficient do exist in the case of interdependent values environments. For frameworks where signals are one-dimensional and a single-crossing property is satisfied, ex post incentive compatibility and efficiency are compatible as mentioned at the beginning of this section.[10]

Still, the equivalence between individual and group ex post incentive compatibility may hold in rather vacuous ways, because there are cases where the only ex post incentive compatible rules lack any interest. But there are other cases where there is a real possibility of making these desiderata compatible in non-trivial ways and thus Theorem 4 opens the door to the existence of full range ex post incentive compatible (and respectful) mechanisms that are ex post group incentive compatible, hence, also Pareto efficient. As evidence of this, in

[9]They show that ex post incentive compatible and ex post stable rules exist in the marriage problem with specific interdependent preferences. A proof that this environment is partially knit is available upon request.

[10]A pioneering paper by Shenker (1993) investigated the connections between individual and group strategy-proof non-bossy social choice rules in economic environments. For a recent reference on efficiency in general environments, see Copic (2017).

each one of the environments in private values mentioned in Subsection 6.1 we refer to a well-known full range mechanism that is ex post group incentive compatible (and respectful). One of them is the family of generalized median voter rules defined on the set of all strict single-peaked preferences (see Moulin, 1980 and our Proposition 3). Another case is provided by the serial dictatorship mechanisms for house allocation problems (see Hylland and Zeckhauser, 1979, Svensson, 1999 and our Proposition 4). A third example is given by veto rules or serial dictators in cases where only two alternatives are at stake and agent's preferences are strict (see Barberà, Berga, and Moreno, 2012a, Manjunath, 2012, Larsson and Svensson, 2006; and Proposition 2 below for two alternatives). A fourth example is the class of peak rules defined by Saporiti (2009) for single-crossing preferences with three alternatives at stake (see Grandmont, 1978 and our Proposition 5).

Also remark that for the case where the mechanism has more than two alternatives on the range, only dictatorship is ex post incentive compatible on the universal set of preferences, by the Gibbard-Satterthwaite theorem (see Gibbard, 1973 and Satterthwaite, 1975). This is an example in which our Theorem 4 also applies, since the universal set of strict preferences is partially knit (see Proposition 2) and dictatorships are ex post group incentive compatible, but we use it here as a warning sign that the implications of Theorem 4, as already explained may or may not be of interest depending on the environments.

An additional implication of ex post group incentive compatibility is that in private values where environments are partially knit, the result in Theorem 4 admits a second reading. This is because ex post incentive compatibility then becomes equivalent to strategy-proofness, since each agent $i$'s preferences depend on $\theta$ only through $\theta_i$. For the same reason, ex post group incentive compatibility becomes equivalent to strong group strategy-proofness.[11] These remarks lead us to the following corollary.

**Corollary 3** *Let* $(\Theta, \mathfrak{R})$ *be any partially knit environment in private values and let* $f$ *be any respectful mechanism in* $(\Theta, \mathfrak{R})$*. Then,* $f$ *is strategy-proof in* $(\Theta, \mathfrak{R})$ *if and only if* $f$ *is strongly group strategy-proof in* $(\Theta, \mathfrak{R})$*.*

This result that applies to private values provides a complementary view of our unifying results in Barberà, Berga, and Moreno (2010, 2016), because partial knitness is not the same domain condition that those that we invoked there, nor is the proof of equivalence the same.

# 5 A discussion about our conditions on environment

Justifying the use of our conditions on environments is clearly needed, since they are certainly involved and abstract. Section 3 already provides results in support of the use of the condition of knitness as one that allows for a neat distinction between environments where ex post

[11] We say that a mechanism $f$ is weakly group manipulable at $\theta \in \Theta$ if there exist a coalition $C \subseteq N$ and $\theta'_C \in \times_{i \in C} \Theta_i$ ($\theta'_i \neq \theta_i$ for any $i \in C$) such that $f(\theta'_C, \theta_{-C}) R_i(\theta_i) f(\theta)$ for all $i \in C$ and $f(\theta'_C, \theta_{-C}) P_j(\theta_j) f(\theta)$ for some $j \in C$. A mechanism $f$ is strongly group strategy-proof in an environment $(\Theta, R)$ if $f$ is not weakly group manipulable at any $\theta \in \Theta$. When the condition is imposed only on singleton coalitions $C = \{i\}$, we say that $f$ is strategy-proof (also called dominant strategy incentive compatible). In words, strategy-proofnes requires that all agents prefer truthtelling at a given type profile $\theta$, whatever all the other agents report.

incentive compatible mechanisms must be constant and others where they can be binary. And, moreover, we have proven that the use of knitness may be a partial guide in search of non-constant mechanisms with range larger than two and holding this property.

We gather below a number of reasons why we came up with knitness and partial knitness, and consider their consequences worth exploring.

Let us first remark that we propose these conditions within a specific framework because the passage from private to interdependent values requires fine adjustments in language and focus. We begin with a very general setting describing choice problems with interdependent values, define preference functions as a mapping from the set of type profiles into the set of preference profiles and environments as pairs formed by a set of type profiles and a preference function. Our conditions are not defined on the domains of mechanisms, which are typically profiles of types, but on environments, in order to stress that the decisive role of the connection between types and preferences via the preference function. Noticing the need to depart from simple restrictions on the domain of types on which mechanisms are defined is part of our modelling contribution.

Our aim is to propose conditions - knitness and partial knitness - that could be applied to a large variety of environments. there are numerous studies using specific conditions in environments (both in private and interdependent values settings), and our purpose is to shed light on the underlying features of the restrictions which contribute to precipitate similar results in seemingly different settings. We certainly cannot claim that our conditions arise from a simple idea, like single-peakedness does. But neither does many others proposed restrictions that have been proved useful to solve different puzzles related to mechanism design. But we would like to explain the nature of our restrictions and how they arise gradually and quite naturally after a careful comparison of the different results we try to approach to each other, and of the techniques that have been used to prove them.

Among the domain restrictions that are proposed in the literature, some are based on whether or not admissible elements in it satisfy a priori determined properties. For example, satisfactory mechanisms may be obtained when preferences are restricted to be separable but not if they must operate over a larger set. And the presumption is that "large enough" domains may lead to impossibilities, while "small enough" ones may admit satisfactory rules. This is not the spirit of the restrictions on environments imposed by knitness or partial knitness. Our conditions are closer to others, like for example, those involving linked domains (Aswal, Chatterji, and Sen, 2003) or semi-single-peaked domains in (Chatterji, Sanver and Sen, 2013). They are better interpreted as requiring that the elements of the environment should be connected to each other, in the sense that one can transform some of its elements into another by means of gradual intermediate changes into ones that also belong to the same environment. No specific element is excluded because of its specific form; much in the same way that single-peakedness does not exclude any specific preferences from being part of a single-peaked domain, but excludes some combinations of preferences that would lead to cycles. Likewise, knitness and partial knitness do not limit a priori what type profiles may be admissible, but are relative to the globality of other type profiles deemed admissible at the same time.

After presenting and proving our main results (Sections 3 and 4) it becomes apparent that the technique of our proofs, and those employed by many other authors, requires the comparison of outcomes that a mechanism would choose along sequences of elements in the environment. Our two conditions are similar requirements regarding the possibility to connect admissible pairs of type profiles through sequences of changes in individual types, whose properties are defined in reference to certain alternatives and through the use of the preference function that is relevant in each environment. They do not exclude in principle any combination of types and preference functions; but once several of such combinations are declared admissible, the restriction limits the characteristics of other potential candidates to be added to the environment while respecting the limitations they impose. Thus, it is not the size of environments, but rather the connections that can be established between the different pairs of type profiles and alternatives. Yet, knitness and partial knitness differs in the pairs of elements to be connected. Whether or not an environment falls in one of these two categories depends on what sequences are considered to be satisfactory, which in turn is determined by the way how the preference function determines the connection between types and preferences. As an example showing that our properties are related to even the most attractive of domain restrictions, in Appendix C we prove that single-peakedness can be interpreted as a form of partial knitness.

# 6 Applications

In this section we present examples of environments satisfying our properties and to which our main results apply. First, we concentrate on well-known frameworks in private values environments. Then, we present and analyze some interdependent values environments in voting and in allocation problems, where our theorems have bite. The proofs of all the results presented in this section are placed in Appendix A.

## 6.1 Private values environments

In private values environments each individual preferences is obtained through the preference function from her own type. That is, changes in $j$'s type do not affect $i$'s preferences if $i \neq j$.

In the following Proposition 1 we state that no private values environment can be knit. Thus, our Theorem 1 has no bite for those environments.

**Proposition 1** *No private values environment* $(\Theta, \mathfrak{R})$ *for which there exist* $\theta_i, \widetilde{\theta}_i \in \Theta_i$ *such that* $R_i(\theta_i) \neq R_i(\widetilde{\theta}_i)$ *for some* $i \in N$ *can be knit.*

The intuition behind the proof of Proposition 1 is simple. Take two pairs of preferences profiles and alternatives such that in one pair, alternative $x$ is strictly preferred to $y$ for some agent and in the other pair, $y$ is the unique best alternative for the same agent $i$. Note that there is no preference relation for agent $i$ that is an $x$-monotonic transform from her preference in the first pair and a $y$-monotonic transform from her preference in the second pair. This together with the fact that in private values other agents' preferences do not affect agent $i$'s preferences implies that the two pairs are not pairwise knit (see Appendix C for an illustration for the case of single-peaked preferences and three alternatives).

However, private values environments can violate partial knitness[12] or satisfy it as the ones defined below. Propositions 2, 3, 4, and 5 state that partial knitness is satisfied by several well-known private values environments. Therefore, the equivalence between ex post individual and group incentive compatibility holds, equivalently Corollary 3 applies. In each one of the applications, we will also mention a relevant full range mechanism which turns out to be Pareto efficient.

To avoid extra notation at this point, we define each one of the environments with detail at the beginning of the proof of the corresponding proposition.

These four private values environments have an additional common characteristic: types and preferences coincide. In particular, for each agent $i$, $R_i(\theta_i) = \theta_i$ and $\Theta_i = \mathcal{R}_i$. Thus, each component of the preference function $\mathfrak{R}$ is the identity and we write the environment simply as the Cartesian product of individual preferences $\times_{i \in N} \mathcal{R}_i$.

We begin by the universal domain of strict preferences.

**Proposition 2** *The Cartesian product of the set of all strict preferences is partially knit.*

In this setting, a dictator[13] is ex post incentive compatible and Pareto efficient since it has full range. Note that a particular case encompassed in Proposition 2 is the one with two alternatives at stake.

Another interesting case is provided by the set of strict single-peaked preferences on a finite set of alternatives. We know that it is not knit by Proposition 1, but as stated in Proposition 3, it is partially knit.

**Proposition 3** *The Cartesian product of the set of all strict single-peaked preferences on a finite set of alternatives is partially knit.*

In this setting, observe that there are also Pareto efficient mechanism: the median rule for an odd number of agents[14] is ex post incentive compatible and Pareto efficient since it has full range.

In the house allocation problem, agents' admissible preferences over their individual assignment are strict. And, again, they define a partially knit environment, as stated in Proposition 4.

**Proposition 4** *The Cartesian product of the set of all preferences in the house allocation problem is partially knit.*[15]

[12] The environment (that is, the domain of preferences) in Barberà, Sonnenschein, and Zhou (1991) is not partially knit since it is well-known that there exist strategy-proof mechanisms that violate (strong) group strategy-proofness.

[13] A mechanism $f$ is *dictatorial* if for every type profile $\theta$, the outcome selected $f(\theta)$ is one of the best alternatives according to her preference $R_i(\theta)$.

[14] A mechanism $f$ is *the median rule* for an odd number of agents if for every type profile $\theta$, the outcome selected $f(\theta)$ is the median of agent's best alternatives according to their preferences $R_i(\theta)$. See Moulin (1980) for further details.

[15] The same result would hold in the one-to-one matching problem where admissible preferences over individual assignments are strict and different for each agent: those of each woman are defined on all men and on herself, while those of each man are defined on all women and himself.

In this setting, a serial dictatorship mechanism is Pareto efficient since it has full range.

The last example we include is one where there are only three alternatives at stake and the set of preferences of each agent is any subset of strict ones. Any domain of such preferences is partially knit.

**Proposition 5** *The Cartesian product of any subset of individual preferences over* $A = \{x, y, z\}$ *is partially knit.*

In this setting, for any strict subset of the universal domain there always exist ex post incentive compatible and full range mechanism.[16] In any case the dictator always does the job.

We remark that the fact that the equivalence holds in some domain of preference profiles does not imply its validity in any of its subdomains, nor viceversa. As mentioned in Section 5 this idea applies for any environment and not just for private values ones. For the latter, when comparing two nested domains of preferences, like the universal domain in Proposition 2 with respect to those considered in Propositions 3, 4, and 5, two things are relevant. First, if the smaller domain is partially knit, the larger domain may not be if some new preference profiles are admissible but can not be connected. Second, if the larger domain is partially knit, that does not imply that the smaller domain is also, because when connecting two pairs of alternatives and preference profiles belonging to the smaller domain one might need a preference profile that belongs to the large domain but not to the small one.

## 6.2 Interdependent values environments

The situations we describe in interdependent and non-private values environments are simple, as examples must be, but chosen to highlight essential contributions to several fields of application: voting and house allocation problems. The examples come in pairs, to show that, with the same sets of type profiles, but depending on the associated preference functions, one can cross the line between positive and negative results. Examples 2 and 3 refer to deliberative juries and are inspired in our reading of Austen-Smith and Feddersen (2006) who build on the classical Condorcet jury problem and add the possibility that agents share (true or false) information. Our second pair of examples, 4 and 5, refer to house allocation problems and are this time inspired by the analysis of Che, Kim, and Kojima (2015), regarding the existence of Pareto efficient and ex-post incentive compatible mechanisms in that context.

These examples are framed in the language we have developed in our paper, and they allow us to clarify several of the points we try to make all along. In particular, we can provide blood and flesh to the general and rather abstract notion of a preference function, by exhibiting how it is defined to fit the particulars of the case at hand.

There are two important differences between these two applications. First, the fact that the type space is either discrete or continuous. For example, Hagen (2019) in a private values setting shows that the necessary and sufficient condition for the existence of non-trivial mechanisms that satisfy group strategy-proofness and symmetry is satisfied if the type set is finite, but not if it is an interval. Second, in the voting application, types are multi-dimensional while in the house allocation application, types are one-dimensional.

[16] As an example, see Berga, Moreno, and Nicolò (2021).

#### 6.2.1 Mechanisms with binary outcome problems with a discrete type space

The two examples in this subsection are motivated by our reading of Austen-Smith and Feddersen (2006) and this is why we use the language of deliberative juries although the example can be applied to other cases that fit the structure of the type space we propose.

**Example 2** A three-person jury $N = \{1,2,3\}$ must decide over two alternatives: whether to acquit ($A$) or to convict ($C$) a defendant under a given mechanism. The defendant is either guilty ($g$) or innocent ($i$). Each juror $j$ gets a signal $s_j = g$ or $s_j = i$.

Jurors's preferences arise from combining the different signals they obtain, according to their bias in favor of acquittal in view of their observed signals and the true signals of others. In this example, jurors are either high-biased ($h$) or low-biased ($l$). High-biased jurors ($h$) prefer to convict if and only if all jurors observe the guilty signal ($s = (g,g,g)$), whereas low-biased ones ($l$) prefer to convict if and only if at least one committee member observed the guilty signal ($s \neq (i,i,i)$).

Each juror $j$'s type is $\theta_j = (b_j, s_j) \in \Theta_j = B \times S$ where $B = \{h,l\}$ and $S = \{g,i\}$. A type profile $\theta \in \Theta = (B \times S)^n$. Let $CA$ denote the preference to convict rather than to acquit and $AC$ be the converse order. The preference function is defined such that for each type profile $\theta \in \Theta$ and for each juror $j \in N$, $R_j(\theta)$ is as follows:

$$R_j\left((b_j,s_j),\theta_{N\setminus\{j\}}\right) = \left\{\begin{array}{l} CA \text{ if either } b_j = h \text{ and } s = (g,g,g) \text{ or } b_j = l \text{ and } s \neq (i,i,i), \\ AC, \text{ otherwise.} \end{array}\right\}$$

The environment $(\Theta, \mathfrak{R})$ in this example is knit (see Proposition 6). Hence we know by Theorem 1 that it will be impossible to design non-constant, ex post incentive compatible, and respectful mechanisms in such framework.

**Proposition 6** *The environment* $(\Theta, \mathfrak{R})$ *in Example 2 is knit.*

We provide the reader with some hints on the techniques that we use to check for our restrictions on environments in this example and subsequent ones.[17]

To check knitness for a particular pair of types and alternatives, $(A,\theta)$ and $(C,\widetilde{\theta})$, we must show that there are passages to a third type profile $\theta'$ which are $A$-satisfactory from $\theta$ and $C$-satisfactory from $\widetilde{\theta}$, respectively.

Consider the following three type profiles, $\theta = (\theta_1,\theta_2,\theta_3) = ((l,g),(h,g),(l,i))$, $\widetilde{\theta} = (\widetilde{\theta}_1,\widetilde{\theta}_2,\widetilde{\theta}_3) = ((l,g),(h,g),(l,g))$ and $\theta' = (\theta'_1,\theta'_2,\theta'_3) = ((l,i),(h,i),(l,i))$. The profiles of preferences they induce are shown in Table 2.

| $R(\theta) = R((l,g),(h,g),(l,i))$ | $R(\widetilde{\theta}) = R((l,g),(h,g),(l,g))$ | $R(\theta') = R((l,i),(h,i),(l,i))$ |
|---|---|---|
| $C$ $A$ $C$ | $C$ $C$ $C$ | $A$ $A$ $A$ |
| $A$ $C$ $A$ | $A$ $A$ $A$ | $C$ $C$ $C$ |

*Table 2*: Agents' preferences induced by $\theta$, $\widetilde{\theta}$, and $\theta'$, respectively.

[17] The reader that finds the following argument useful to better understand our condition may also find a similar one regarding partial knitness in the text preceding the proof of Proposition 7 in Appendix B.

As shown in Table 3, it is possible to sequentially move from $\theta$ to $\theta'$ by successively changing, one by one, the type of the agents as follows. First, agent 1 from $(l,g)$ to $(h,i)$, then agent 2 from $(h,g)$ to $(h,i)$ and finally agent 1 from $(h,i)$ to $(l,i)$. According to our notation, $S=((h,i),(h,i),(l,i))$ and $I(S)=\{1,2,1\}$. Likewise, as shown in Table 4, we can move from $\widetilde{\theta}$ to $\theta'$ by successively changing, one by one, the type of some agents. First, agent 1, then agent 3 and finally agent 2, all from signal $g$ to $i$, while their $b$'s remain fixed. That is, $\widetilde{S}=((l,i),(l,i),(h,i))$ and $I(\widetilde{S})=\{1,3,2\}$. In Table 3, alternative $A$ either does not change its relative position (an $A$-reshuffling), or improves it (an $A$-monotonic transform). Similarly, in Table 4, the same requirements are satisfied but this time for alternative $C$.

| $R(\theta)$ | $R^1(\theta,S)$ | $R^2(\theta,S)$ | $R^3(\theta,S)=R(\theta')$ |
|---|---|---|---|
| $R((l,g),(h,g),(l,i))$ | $R((\mathbf{h},\mathbf{i}),(h,g),(l,i))$ | $R((h,i),(h,\mathbf{i}),(l,i))$ | $R((\mathbf{l},i),(h,i),(l,i))$ |
| $C\ \ A\ \ C$ | $A\ \ A\ \ C$ | $A\ \ A\ \ A$ | $A\ \ A\ \ A$ |
| $A\ \ C\ \ A$ | $C\ \ C\ \ A$ | $C\ \ C\ \ C$ | $C\ \ C\ \ C$ |

*Table 3*: Induced agents' preferences given the specified type changes from $\theta$ to $\theta'$.

| $R(\widetilde{\theta})$ | $R^1(\widetilde{\theta},\widetilde{S})$ | $R^2(\widetilde{\theta},\widetilde{S})$ | $R^3(\widetilde{\theta},\widetilde{S})=R(\theta')$ |
|---|---|---|---|
| $R((l,g),(h,g),(l,g))$ | $R((l,\mathbf{i}),(h,g),(l,g))$ | $R((l,i),(h,g),(l,\mathbf{i}))$ | $R((l,i),(h,\mathbf{i}),(l,i))$ |
| $C\ \ C\ \ C$ | $C\ \ A\ \ C$ | $C\ \ A\ \ C$ | $A\ \ A\ \ A$ |
| $A\ \ A\ \ A$ | $A\ \ C\ \ A$ | $A\ \ C\ \ A$ | $C\ \ C\ \ C$ |

*Table 4*: Induced agents' preferences given the specified type changes from $\widetilde{\theta}$ to $\theta'$.

**Example 3** Consider the framework of Example 2 and change the jurors' attitude to convict versus acquit as follows. Each juror may now be either unswerving or median. Unswerving jurors ($u$) prefer to convict if and only if they have observed the guilty sign and at least another juror has also observed such a signal. Median jurors ($m$) again prefer to convict under the same circumstances but also if the other two jurors observe the guilty signal.

For instance, if juror 1 is unswerving she will prefer to convict if either $(g,g,g)$, $(g,g,i)$, or $(g,i,g)$ but if juror 2 is unswerving she will convict if either $(g,g,g)$, $(g,g,i)$, or $(i,g,g)$). Yet being median is the same for both agents, they will prefer to convict if either $(g,g,g)$, $(g,g,i)$, $(g,i,g)$, or $(i,g,g)$.

Each juror $j$'s type is $\theta_j=(b_j,s_j)\in\Theta_j=B\times S$ where $B=\{u,m\}$ and $S=\{g,i\}$. A type profile $\theta\in\Theta=(B\times S)^n$. The preference function is defined such that for each type profile $\theta$ and for each juror $j\in N$, $R_j(\theta)$ is as follows:

$$R_j\left((b_j,s_j),\theta_{N\setminus\{j\}}\right)=\left\{\begin{array}{ll} CA & \text{if either } b_j=u,\ s_j=g \text{ and } s_l=g \text{ for some } l\neq j, \\ & \text{or } b_j=m \text{ and } \#\{l\in N: s_l=g\}\geq 2, \text{ and} \\ AC & \text{otherwise.}\end{array}\right\}$$

This environment $(\Theta,\mathfrak{R})$ is partially knit (see Proposition 7) but not knit.

**Proposition 7** *The environment* $(\Theta, \mathfrak{R})$ *in Example 3 is partially knit.*

To show that it is not knit, we present a family of mechanisms, the quota rules, that are non-constant, respectful, and ex post incentive compatible in $(\Theta, \mathfrak{R})$ which is stated in Remark 2.

Let $q \in \{1, 2, 3\}$. A *voting by quota q* mechanism, $f$, chooses $C$ for a type profile $\theta$ if and only if at least $q$ agents have induced preferences from $\theta$ such that $C$ is preferred to $A$.[18] Formally, for each type profile $\theta = (b, s) \in \Theta$,

$$f(\theta) = C \text{ if and only if } \# \{i \in N : R_i(\theta) = CA\} \geq q.$$

**Remark 2** *A voting by quota q mechanism is non-constant, ex post incentive compatible, and respectful in the environment* $(\Theta, \mathfrak{R})$ *in Example 3.*

Now, Theorem 4 will ensure that these and other mechanisms that we may know to be ex post incentive compatible for our example will also be ex post group incentive compatible (therefore, Pareto efficient on the range) since the environment is partially knit. Thus, Pareto efficiency is satisfied in this example because the mechanism has full range.

#### 6.2.2 Mechanisms with binary outcome problems with a continuous type space

The two examples in this subsection are motivated by our reading of Che, Kim, and Kojima (2015) and this is why we use the language of house allocation problems although the example can be applied to other cases that fit the structure of the type space we propose.

**Example 4** Let $N = \{1, 2\}$ be a set of agents, $O = \{a, c\}$ be a set of objects. Each agent must be assigned one and only one object. Thus, the set of alternatives is $A = \{x = (a, c), z = (c, a)\}$, where the first component refers to the object that agent 1 gets. There is no money in this economy.

The type $\theta_i \in \Theta_i$ of each agent $i$ is given by a signal $\theta_i$ in $\Theta_i = [0, 1]$. Each individual $i \in N$ is endowed with a given auxiliary function $g_i : \Theta \to \mathbb{R}$ *strictly increasing* in both signals. The preference function $\mathfrak{R}$ is such that for each agent $i \in N$ and for each type profile $\theta \in \Theta = [0, 1] \times [0, 1]$, $R_i(\theta)$ is as follows: $x$ is at least as good as $z$ if and only if $g_i(\theta) \geq 0$.

The environment in Example 4 is knit (see Proposition 8). Therefore by Theorem 1 only constant mechanisms can be ex post incentive compatible and respectful in this context.

**Proposition 8** *The environment* $(\Theta, \mathfrak{R})$ *in Example 4 is knit.*

**Example 5** We consider the framework of Example 4, except that we change agents' preference functions to be induced by $g_1(\theta) = \min\left(median\left\{\frac{1}{4}, \theta_1, \theta_1, \theta_2\right\}\right) - \frac{1}{4}$ and $g_2(\theta) = \min\left(median\left\{\frac{1}{4}, \theta_2, \theta_2, \theta_1\right\}\right) - \frac{1}{4}$, respectively. That is, for each agent $i \in N$ and for each type profile $\theta \in \Theta$, $R_i(\theta)$ is as follows: $x$ is at least as good as $z$ if and only if $g_i(\theta) \geq 0$.

The main but significant difference between this example and the preceding one is that now the functions $g_i$ are just weakly increasing.

Like in Example 3 above, the environment in this example is partially knit (see Proposition 9) but not knit.

[18] See Austen-Smith and Feddersen (2006) and Barberà and Jackson (2004) for papers where these rules are analyzed.

**Proposition 9** *The environment* $(\Theta, \mathfrak{R})$ *in Example 5 is partially knit.*

To prove non-knitness, we consider the veto mechanisms defined below. Before introducing them we need the following definition: consider a partition of the type (signal) space and a useful graphical representation of it which is similar to the one defined in Che, Kim, and Kojima (2015).

Let $\{S_{ac}, S_{ca}, S_{aa}, S_{cc}, S^0\}$ be the partition of $\Theta$ where:
$S^0$ is the set of type profiles for which both agents are indifferent between $a$ and $c$,
$S_{ac}$ is the set of type profiles for which agent 1 prefers $a$ to $c$, agent 2 prefers $c$ to $a$, and the preferences are strict for at least one agent,
$S_{ca}$ is equally defined after changing the roles of $c$ and $a$,
$S_{aa}$ is the set of type profiles for which both agents prefer $a$ to $c$, and
$S_{cc}$ is equally defined after changing the roles of $c$ and $a$.

In terms of alternatives, when the type profiles are in $S_{ac}$ both agents prefer $x$ to $z$, when they are in $S_{ca}$ both prefer $z$ to $x$, in $S_{aa}$, 1 prefers $x$ over $z$ and 2 prefers $z$ over $x$, in $S_{cc}$, 1 prefers $z$ over $x$ and 2 prefers $x$ over $z$, and in $S^0$ both are indifferent between $x$ and $z$.

Figure 1 provides a generic representation of these sets whose frontiers correspond to the pairs of signals leading to agents' indifference curves over alternatives: $\{\theta \in \Theta = [0,1] \times [0,1] : xI_i(\theta)y\}$. Since we have assumed that $g_i$ is strictly increasing in both types, agents' indifference curves are strictly decreasing. In this figure we represent a situation where all the elements in the partition of $\Theta$ are non-empty and moreover the two agents' indifference curves have an interior intersection.[19]

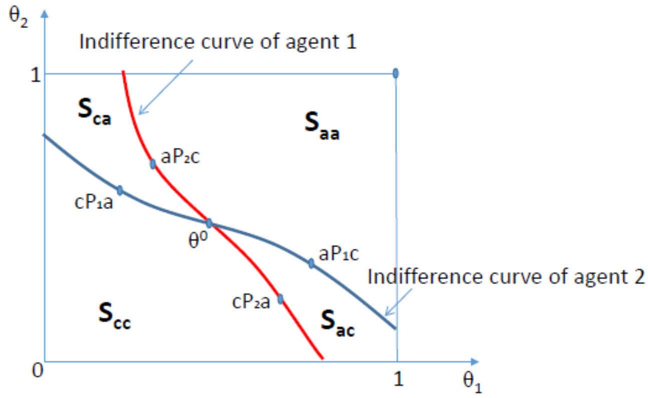


Figure 1. Examples of the partition of S in Example 4.

Now we say that a mechanism $f_{veto\ x}$ is a *veto rule for* $x$ if for any type profile the outcome is agent 1's best alternative when it is unique, and it is agent 2's best alternative otherwise. Formally, for $\theta \in \Theta = [0,1] \times [0,1]$,

$$f_{veto\ x}(\theta) = \left\{ \begin{array}{c} x = (a,c) \text{ if } \theta \in S_{ca}, \text{ and} \\ z = (c,a) \text{ if } \theta \in S_{aa} \cup S_{ac} \cup S_{cc} \cup S^0 \end{array} \right\}.$$

[19] In general, some elements in the partition could be empty and the intersection of the two curves may no be interior. Moreover, although in all pictures corresponding to this example the indifference curves only intersect once, our formal arguments apply to the multiple intersection case.

In view of Theorem 1 the existence of these non-constant, ex post incentive compatible, and respectful mechanisms implies that the environment is no longer knit. In Remark 3 we show that veto rules satisfy the three properties.

**Remark 3** *$f_{veto,x}$ is non-constant, ex post incentive compatible, and respectful in the environment $(\Theta, \mathfrak{R})$ in Example 5.*

Now, Theorem 4 will ensure that these and other mechanisms that we may know to be ex post incentive compatible for our example will also be ex post group incentive compatible (therefore, Pareto efficient on the range) since the environment is partially knit. Thus, Pareto efficiency is obtained in the example since the mechanism has full range.

# 7 Further comments

We have approached issues about the design of ex post incentive compatibility in a purely ordinal framework, and this makes it difficult to compare our results with the conclusions of other works, like those on auction design where money metrics play an important role. Yet we hope that our methodological proposals and our emphasis on the role of domain restrictions may be useful and inspire some further research.

In particular, having proved that when the preferences of agents are strict, non-constant ex post incentive compatible mechanisms on binary ranges may be designed if and only if they operate on environments that are not knit, we would like to pursue two lines of study that are only suggested but not fully solved in the present work.

One is to follow after Theorem 2 and to study relaxations of knitness that would be sufficient to allow for mechanisms to satisfy this requirement on incentives for larger and eventually full ranges. Another is to continue our search for conditions under which our Theorem 4, about domains allowing for ex post group incentive compatibility, could be finessed to guarantee the existence of attractive full range mechanisms, accommodating incentive compatibility and efficiency properties simultaneously. Progress in these two directions might approach the conclusions we have gotten in our Theorems 1 and 4.

Finally, we mention another line of research that we would find useful, not only for our present purposes but also for the general treatment of mechanism design problems. To keep looking for relaxations of technical properties that most papers need when allowing for indifferences in the preferences of agents, starting with the original condition of non-bossiness. Our condition of respectfulness, although less demanding than other requirements in the same vein, it is still an obstacle for the full generality of our results, although these become transparent and conclusive in the case of strict domains.

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# Appendix A. Proofs of the results in the Applications' section

## Private values

**Proof of Proposition 1.** Let $i \in N$ and $\theta_i, \widetilde{\theta}_i \in \Theta_i$, $\theta_i \neq \widetilde{\theta}_i$ be such that $R_i(\theta_i) \neq R_i(\widetilde{\theta}_i)$. That is, $R_i(\theta_i, \theta_{N\setminus\{i\}}) \neq R_i(\widetilde{\theta}_i, \theta_{N\setminus\{i\}})$ for all $\theta_{N\setminus\{i\}} \in \times_{j\in N\setminus\{i\}}\Theta_j$ since $(\Theta, \mathfrak{R})$ is a private values environment. Then, there will be a pair of alternatives, say $x$ and $z$, such that $xP_i(\theta_i)z$ and $zR_i(\widetilde{\theta}_i)x$ (otherwise, for $\theta_i, \widetilde{\theta}_i \in \Theta_i$, $R_i(\theta_i) = R_i(\widetilde{\theta}_i)$). To show that the environment $(\Theta, \mathfrak{R})$ is not knit, we prove that the two pairs $(x, (\theta_i, \theta_{N\setminus\{i\}}))$, $(z, (\widetilde{\theta}_i, \theta_{N\setminus\{i\}}))$, whatever $\theta_{N\setminus\{i\}}$, are not pairwise knit. That is, there does not exist any $\theta'$, $S$, and $\widetilde{S}$ such that the passage from $\theta$ to $\theta'$ through $S$ be $x$-satisfactory and the passage from $\widetilde{\theta}$ to $\theta'$ through $\widetilde{S}$ be $z$-satisfactory. We prove it by contradiction. Suppose otherwise that there exist $\theta^*$, $S^*$, $\widetilde{S}^*$, such that the passages $\{m^h(\theta, S^*)\}_{h=0}^{t_{S^*}}$ and $\left\{m^h(\theta, \widetilde{S}^*)\right\}_{h=0}^{t_{\widetilde{S}^*}}$ from $\theta$ to $\theta^*$ through $S^*$ and $\widetilde{\theta}$ to $\theta^*$ through $\widetilde{S}^*$ are $x$ and $z$-satisfactory, respectively.

Since we are in a private values environment, changes in the type of agent $j$ never affect the induced preferences of other agents, in particular never affect $i$'s induced preferences if $j \neq i$. Moreover, we know that $xP_i(\theta_i, \theta_{N\setminus\{i\}})z$ and $zR_i(\widetilde{\theta}_i, \theta_{N\setminus\{i\}})x$. These two observations imply that agent $i$ must belong to $I(S^*) \cup I(\widetilde{S}^*)$. That is, $i$ will appear in at least one of these two sequences.

We concentrate on the steps of the passage where agent $i$ changes her type and we show that there is no $\theta^*$ compatible with $x$-satisfactory and $z$-satisfactory passages from $\theta$ to $\theta^*$ and from $\widetilde{\theta}$ to $\theta^*$.

Without loss of generality, by the remark just after Definition 8, we can assume that all types of agent $i$ in $S^*$ and $\widetilde{S}^*$ appear in the first positions in these sequences. Let's define $I_{S^*,i} \equiv \{h \in \{1, 2, ..., i_{S^*}\} : i(S^*, h) = i\}$ and $I_{\widetilde{S}^*,i} = \left\{h \in \{1, 2, ..., i_{\widetilde{S}^*}\} : i(\widetilde{S}^*, h) = i\right\}$.

Take $1 \in I_{S^*,i}$. Since $R_i^1(\theta, S^*)$ is an $x$-monotonic transform of $R_i(\theta_i, \theta_{N\setminus\{i\}})$, we have that $xP_i(m_i^1(\theta, S^*))z$. By repeating the same argument for each $h \in I_{S^*,i}$ we finally obtain that $xP_i(m_i^{i_{S^*}}(\theta, S^*))z$ where $m_i^{i_{S^*}}(\theta, S^*) = \theta_i^*$.

Now, take $1 \in I_{\widetilde{S}^*,i}$. Since $R_i^1(\widetilde{\theta}^*, \widetilde{S}^*)$ is a $z$-monotonic transform of $R_i(\widetilde{\theta}_i^*, , \theta_{N\setminus\{i\}})$, we have that $zR_i(m_i^1(\widetilde{\theta}^*, \widetilde{S}^*))x$. By repeating the same argument for each $h \in I_{\widetilde{S}^*,i}$ we finally obtain that $zR_i(m_i^{i_{\widetilde{S}^*}}(\theta, \widetilde{S}^*))z$ where $m_i^{i_{\widetilde{S}^*}}(\theta, \widetilde{S}^*) = \theta_i^*$.

As mentioned above, changes in types of agents different from $i$ will not change agent $i$'s preferences. Thus, we have obtained the desired contradiction. On the one hand that $xP_i(\theta^*)z$ and on the other hand, that $zR_i(\theta^*)x$. ■

For the private values environments in Propositions 2, 3, 4, and 5, the following two relevant observations hold and are used in their proofs: types are preferences in these cases, that is, $\theta_i = R_i \in \mathcal{R}_i = \Theta_i$ for each $i \in N$. Moreover, changes in $j$'s preferences do not affect $i$'s preferences if $i \neq j$.

**Proof of Proposition 2.** Let $\mathcal{U}$ denote the universal set of strict preferences. Thus, $\mathcal{R}_i = \mathcal{U}$. To prove partial knitness, take any $(x, R)$, $(z, \widetilde{R}) \in A \times \mathcal{U}^n$ such that $\overline{C}(R, z, x) =$

$C(R,z,x) \neq \varnothing$, $\#C(R,z,x) \geq 2$, and $\widetilde{R}_j = R_j$ for all $j \in N \backslash C(R,z,x)$. Without loss of generality, let $C(R,z,x) = \{1,2,...,c\}$ where $c$ also denotes its cardinality. Now, we must show that $(x,R)$, $(z,\widetilde{R})$ are pairwise knit. To do that, we construct $S$, $\widetilde{S}$ and $R'$ satisfying the condition in pairwise knitness.

For each $R_i \in \mathcal{U}$, let us denote by $R_i^z$ the preference obtained by lifting $z$ to the first position and keep the relative position of all other alternatives.

Now, start from $R$ and define $S = \{R_1^z, R_2^z, ..., R_c^z\}$ where $t_S = c$. Note that for each $h \in \{1,...,c\}$, $R_j^h(R,S) = R_j^{h-1}(R,S)$ for all $j \in N \backslash i(S,h)$ and $R_{i(S,h)}^h(R,S) = R_{i(S,h)}^z \in \mathcal{U}$. That is, for all $i$, $R_i^h(R,S)$ is an $x$-reshuffling of $i$'s previous preferences $R_i^{h-1}(R,S)$. Then, $R' = R^c(R,S) = (R_{C(R,z,x)}^z, R_{N \backslash C(R,z,x)}) \in \mathcal{U}^n$.

Now, start from $\widetilde{R}$ and define $\widetilde{S} = \left\{\widetilde{R}_1^z, \widetilde{R}_2^z, ..., \widetilde{R}_c^z, R_1^z, R_2^z, ..., R_c^z\right\}$ where $t_{\widetilde{S}} = 2c$. For each $h \in \{1,...,c\}$, $R_j^h(\widetilde{R},\widetilde{S}) = R_j^{h-1}(\widetilde{R},\widetilde{S})$ for all $j \in N \backslash i(\widetilde{S},h)$ and $R_{i(\widetilde{S},h)}^h(\widetilde{R},\widetilde{S}) = \widetilde{R}_{i(\widetilde{S},h)}^z$. That is, for all $i$, $R_i^h(\widetilde{R},\widetilde{S})$ is a $z$-monotonic transform or a $z$-reshuffling (if $z$ was already the top or does not change preferences) of $i$'s previous preferences $R_i^{h-1}(\widetilde{R},\widetilde{S})$. Moreover, for $h \in \{c+1,...,2c\}$, $R_j^h(\widetilde{R},\widetilde{S}) = R_j^{h-1}(\widetilde{R},\widetilde{S})$ for all $j \in N \backslash i(\widetilde{S},h)$ and $R_{i(\widetilde{S},h)}^h(\widetilde{R},\widetilde{S}) = R_{i(\widetilde{S},h)}^z$, which for each agent is a $z$-reshuffling of her previous preferences $\widetilde{R}_i^z$. Then, $R' = R^{2c}(\widetilde{R},\widetilde{S}) = (R_{C(R,z,x)}^z, R_{N \backslash C(R,z,x)})$. ■

**Proof of Proposition 3.** Let $A$ be a finite and ordered set of alternatives in $\mathbb{R}$, the real line. For all $i \in N$, let $\mathcal{R}_i = \mathcal{S}$ be the set of strict single-peaked preferences on $A$ according to the established real numbers order. We introduce some notation: given $R_j \in \mathcal{S}$, $p(R_j)$ denotes the peak, that is, the best alternative, of $R_j$ in $A$. Let $\overline{L}(R_i,x) = \{y \in A : xP_iy\}$ be the *strict lower contour set of* $R_i$ *at* $x$. Given $R_j \in \mathcal{S}$ and $x \in A$, define $r(R_j,x)$ as the first alternative in $\overline{L}(R_j,x)$ in the opposite side of alternative $x$ with respect to $p(R_j)$.

To prove partial knitness, take any $(x,R), (z,\widetilde{R}) \in A \times \mathcal{S}^n$ such that $\overline{C}(R,z,x) = C(R,z,x) \neq \varnothing$, $\#C(R,z,x) \geq 2$, and $\widetilde{R}_j = R_j$ for all $j \in N \backslash C(R,z,x)$ and show that $(x,R)$, $(z,\widetilde{R})$ are pairwise knit. Without loss of generality, let $x < z$, which implies that $p(R_j) > x$. Also without loss of generality, let $C(R,z,x) = \{1,2,...,c\}$ where $c$ denotes its cardinality. Now define $I(S) = I(\widetilde{S}) = C(R,z,x) = \{1,2,...,c\}$ and construct for each agent $j \in \{1,2,...,c\}$, $R'_j$ depending on the cases below.

Take any $j \in C(R,z,x)$ and consider the following cases.

<u>Case 1</u>. $\widetilde{R}_j$ is such that $x\widetilde{P}_jz$. Take $R'_j \in \mathcal{S}$ such that $p(R'_j) \in [x,z)$, $r(R_j,x) = z$, and $zP'_jy$ for all $y < x$. Notice that such $R'_j$ exists, and the two following set inclusions hold: $\overline{L}(R_j,x) \subseteq \overline{L}(R'_j,x)$, $\overline{L}(\widetilde{R}_j,z) \subseteq \overline{L}(R'_j,z)$. Thus, $R'_j$ is both an $x$-monotonic transform *of* $R_j$ and a $z$-monotonic transform of $\widetilde{R}_j$ (observe that with strict preferences, the above inclusion of strict lower contour sets is equivalent to Definition 1).

<u>Case 2</u>. $\widetilde{R}_j$ is such that $z\widetilde{P}_jx$. Consider several subcases.

<u>Case 2.1</u>. $\overline{L}(R_j,x) \subseteq \overline{L}(\widetilde{R}_j,x)$. Let $R'_j = \widetilde{R}_j$ and observe that $R'_j$ is an $x$-monotonic transform of $R_j$ (obviously, $R'_j$ is a $z$-monotonic transform of $\widetilde{R}_j$ since $R'_j = \widetilde{R}_j$).

<u>Case 2.2</u>. $\overline{L}(\widetilde{R}_j,x) \subsetneq \overline{L}(R_j,x)$. We distinguish additional subcases which require different definitions of $R'_j$.

*Case 2.2.1*. $\overline{L}(\widetilde{R}_j, x) \subsetneqq \overline{L}(R_j, x)$ and $\overline{L}(\widetilde{R}_j, z) \subseteq \overline{L}(R_j, z)$. Let $R'_j = R_j$ and observe that $R'_j$ is an $x$-monotonic transform of $R_j$ (obviously since $R'_j = R_j$) and $R'_j$ is also a $z$-monotonic transform of $\widetilde{R}_j$.

*Case 2.2.2*. $\overline{L}(\widetilde{R}_j, x) \subsetneqq \overline{L}(R_j, x)$ *and* $\overline{L}(R_j, z) \subsetneqq \overline{L}(\widetilde{R}_j, z)$. This implies that either *(a)* $p(R_j), p(\widetilde{R}_j) \in (x, z)$ or else *(b)* $p(R_j), p(\widetilde{R}_j) > z$.

If *(a)* holds, then let $R'_j$ be such that $p(R'_j) \in \Big[\min\{p(R_j), p(\widetilde{R}_j)\}, \max\{p(R_j), p(\widetilde{R}_j)\}\Big]$, $r(R'_j, x) = r(R_j, x)$ and $r(R'_j, z) \geq r(\widetilde{R}_j, z)$. By definition of single-peakedness, such preferences $R'_j$ exists.

If *(b)* holds, then let $R'_j$ be such that $p(R'_j) \in \Big[z, \min\{p(R_j), p(\widetilde{R}_j)\}\Big]$, $r(R'_j, x) \leq r(R_j, x)$ and $r(R'_j, z) \leq r(\widetilde{R}_j, z)$. By definition of single-peakedness, such preferences $R'_j$ exists.

Then, observe that $R'_j$ defined in *(a)* and *(b)* is both an $x$-monotonic transform of $R_j$ and a $z$-monotonic transform of $\widetilde{R}_j$ since $\overline{L}(R_j, x) \subseteq \overline{L}(R'_j, x)$ and $\overline{L}(\widetilde{R}_j, z) \subseteq \overline{L}(R'_j, z)$ hold.

*Case 2.2.3*: $\overline{L}(\widetilde{R}_j, x) \subsetneqq \overline{L}(R_j, x)$ and $z \in \Big(\min\{p(R_j), p(\widetilde{R}_j)\}, \max\{p(R_j), p(\widetilde{R}_j)\}\Big)$. Assume that $p(R_j) < z < p(\widetilde{R}_j)$, otherwise, a similar argument would work.

This implies that either *(a)* $r(R_j, x) \in \Big(z, p(\widetilde{R}_j)\Big]$ or *(b)* $r(R_j, x) \in \Big(p(\widetilde{R}_j), r(\widetilde{R}_j, x)\Big)$ holds.

If *(a)* holds, then let $R'_j$ be such that $p(R'_j) \in [z, r(R_j, x))$, $r(R'_j, x) \leq r(R_j, x)$ and $r(R'_j, z) \leq r(\widetilde{R}_j, z)$. By definition of single-peakedness, such preferences $R'_j$ exists.

If *(b)* holds, then let $R'_j$ be such that $p(R'_j) \in \Big[z, \min\{r(R_j, x), r(\widetilde{R}_j, z)\}\Big)$, $r(R'_j, x) \leq r(R_j, x)$ and $r(R'_j, z) \leq r(\widetilde{R}_j, z)$.

Then, observe that $R'_j$ in *(a)* and *(b)* is both an $x$-monotonic transform of $R_j$ and a $z$-monotonic transform of $\widetilde{R}_j$ since $\overline{L}(R_j, x) \subseteq \overline{L}(R'_j, x)$ and $\overline{L}(\widetilde{R}_j, z) \subseteq \overline{L}(R'_j, z)$ hold.

Finally, for each $j \in C(R, z, x)$ we repeat the same argument. ■

**Proof of Proposition 4.** The proof follows the same argument as the one in Proposition 2, given that agents have all possible strict preferences over individual assignments and preferences are selfish. As in Barberà, Berga, and Moreno (2016), just note that although preferences over individual assignments are strict, preferences over alternatives allow for indifferences, by selfishness: all alternatives with the same individual assignment are indifferent for such individual agent. Thus, in the case of the house allocation problem $C(R, z, x) \supseteq \overline{C}(R, z, x)$ holds and $R_i^z$ are the preferences obtained by lifting $z$ and also all alternatives with the same individual assignment $z_i$ to the first position and keep the relative position of all other alternatives. ■

**Proof of Proposition 5.** Let $A = \{x, y, z\}$ be the set of alternatives. Let $\widetilde{\mathcal{L}}$ be the set of all strict preferences on $A$ and for each agent $i \in N$, let $\mathcal{D}_i \subseteq \widetilde{\mathcal{L}}$ be the set of $i$'s preferences. It is worth noting that for each $i \in N$ and each pair of alternatives $a, b \in A$ there exist at most three individual preferences in $\mathcal{D}_i$ such that $aP_ib$, two of them with $b$ as the worst alternative and another one with $b$ in the middle position. To show that $\times_{i\in N}\mathcal{D}_i$ is partially knit, take any pair $(x, R)$, $(z, \widetilde{R}) \in A \times (\times_{i\in N}\mathcal{D}_i)$ such that $\overline{C}(R, z, x) = C(R, z, x) \neq \varnothing$, $\#C(R, z, x) \geq 2$, and $\widetilde{R}_j = R_j$ for all $j \in N \backslash C(R, z, x)$

and show that $(x, R)$, $(z, \widetilde{R})$ are pairwise knit. Let $S(R)$ and $\widetilde{S}(R)$ be the partition of $\overline{C}(R, z, x)$ such that $S(R) = \{i \in \overline{C}(R, z, x) : x \text{ is bottom according to } R_i\}$ and $\widetilde{S}(R) = \{i \in \overline{C}(R, z, x) : x \text{ is second according to } R_i\}$ (well-defined by the note above).
Let $R_i' = \widetilde{R}_i$ for each $i \in S(R)$, let $S = S(R)$, and observe that for each $i \in S(R)$ and each possible $\widetilde{R}_i$, $R_i' = \widetilde{R}_i$ is an $x$-monotonic transform of $R_i$ since $x$ is bottom of $R_i$.
Let $R_i' = R_i$ for each $i \in \widetilde{S}(R)$, let $\widetilde{S} = \widetilde{S}(R)$, and observe that for each $i \in \widetilde{S}(R)$ and each possible $\widetilde{R}_i$, $R_i' = R_i$ is an $z$-monotonic transform of $\widetilde{R}_i$ since $z$ is top of $R_i$. ■

## Mechanisms with binary outcome problems with a discrete type space

**Proposition 6** *The environment* $(\Theta, \mathfrak{R})$ *in Example 2 is knit.*

**Proof of Proposition 6.** To prove knitness we just need to combine the following two results.
(1) Consider a pair formed by $(A, \theta)$ for any $\theta \in \Theta$ where $\theta_j = (b_j, s_j)$ for each $j \in N$. Let $\theta' \in \Theta$ be such that $\theta_1' = (l, i)$ and $\theta_j' = (h, i)$ for any $j \in N \backslash \{1\}$. We now define the sequence $S$ to sequentially go from type profile $\theta$ to type profile $\theta'$ by successively changing the type of the agents in $S$ while preserving $A$-satisfactoriness. First change, one by one and in any order, agents' signals from $s_j \neq i$ to $i$. By definition of $l$ and $h$, in each of the above changes, the induced preferences of the agent changing her type is an $A$-monotonic transform of her previous preferences (sometimes an $A$-reshuffling).
Observe that by definition of the preference functions, the following condition is satisfied: if $\widehat{s}_j = i$ for all $j \in N$, all jurors prefer $A$ to $C$ for any $\widehat{b}_j \in B$.
We now change, one by one and in any order, each agent's $b_j \neq h$ from $b_j$ to $h$ for any $j \in N \backslash \{1\}$ and from $b_1 \neq l$ to $l$ in the case of agent 1. By the observation made just above, in each of these changes, the induced preferences of each agent is the same and therefore they are an $A$-reshuffling of their previous preferences. Then, we have defined $S$ such that $\theta$ leads to $\theta'$ through $S$ and the passage from $\theta$ to $\theta'$ is $A$-satisfactory.
(2) Consider a pair $(C, \theta)$ for any $\theta \in \Theta$ where $\theta_j = (b_j, s_j)$ for each $j \in N$. We now define the sequence $S$ to go from type profile $\theta$ to $\theta'$ above by successively changing the type of the agents in $S$ while preserving $C$-satisfactoriness. First change, one by one and in any order, agents from $s_j \neq g$ to $g$. By definition of $l$ and $h$, in each of the above changes, the induced preferences of the agent changing her type is a $C$-monotonic transform of her previous preferences (sometimes a $C$-reshuffling).
Observe that by definition of the preference function, the following property is satisfied: if $\widehat{s}_j = g$ for all $j \in N$, all jurors prefer $C$ to $A$ for any $\widehat{b}_j \in B$.
We now change one by one, and in any order, each agent's $b_j \neq h$ from $b_j$ to $h$ for any $j \in N \backslash \{1\}$ and from $b_1 \neq l$ to $l$ in the case of agent 1. By the observation made just above, in each of these steps, the preferences of the agents stay the same and therefore they are a $C$-reshuffling of their previous ones. After that, we change the signal of the agent 1 from $g$ to $i$. This implies that the preferences of agent 1 remain identical, but those of all others go from $C$ preferred to $A$, to $A$ preferred to $C$, given that $b_j = h$ for any $j \in N \backslash \{1\}$. Finally, we change the type of the rest of the agents one by one from $g$ to $i$. In each one of these

steps the preferences of the agent that moves is still $A$ preferred to $C$. The passage from $\theta$ to $\theta'$ is $C$-satisfactory by construction. ■

Before engaging in the proof that the environment in Example 3 is partially knit (see Proposition 7), we develop the argument for a particular example as mentioned in Footnote 14.

Consider a particular pair of types and alternatives, $(A, \theta)$ and $(C, \widetilde{\theta})$ where $\theta = ((u,g),(u,i),(m,g))$ and $\widetilde{\theta} = ((m,i),(u,i),(u,g))$. Let $\theta' = ((m,i),(u,i),(m,g))$. The profiles of preferences they induce are shown in Table 5.

| $R(\theta) = R((u,g),(u,i),(m,g))$ | $R(\widetilde{\theta}) = R((m,i),(u,i),(u,g))$ | $R(\theta') = R((m,i),(u,i),(m,g))$ |
|---|---|---|
| $C$ $A$ $C$ | $A$ $A$ $A$ | $A$ $A$ $A$ |
| $A$ $C$ $A$ | $C$ $C$ $C$ | $C$ $C$ $C$ |

*Table 5*: Agents' preferences induced by $\theta$, $\widetilde{\theta}$, and $\theta'$, respectively.

We can check that $\overline{C}(\theta, C, A) = C(\theta, C, A) = \{1, 3\}$ and $\widetilde{\theta}_2 = \theta_2$ (that is, requirements in Definition 11 are satisfied). As shown in Table 6 below, it is possible to move from $\theta$ to $\theta'$ by successively changing, one by one, the type of the agents. In this case, agent 1 from $(u, g)$ to $(m, i)$. According to our notation, $I(S) = \{1\}$. Likewise, as shown in Table 7 below, we can move from $\widetilde{\theta}$ to $\theta'$ by successively changing, one by one, the type of some agents. In this case, agent 3 from $(u, g)$ to $(m, g)$, that is, $I(\widetilde{S}) = \{3\}$. In Table 6, note that the preferences $R_1(\theta')$ of agent 1 are an $A$-monotonic transform of her previous ones, which also involve a change of those for agent 3. Similarly, notice that the preferences $R_3(\theta')$ of 3 in Table 7 are a $C$-reshuffling of her previous ones.

| $R(\theta) = R,((ug),(u,i),(m,g))$ | $R(\theta') = R((\mathbf{m},\mathbf{i}),(u,i),(m,g))$ |
|---|---|
| $C$ $A$ $C$ | $A$ $A$ $A$ |
| $A$ $C$ $A$ | $C$ $C$ $C$ |

*Table 6*: Induced agents' preferences given the specified type changes from $\theta$ to $\theta'$.

| $R(\widetilde{\theta}) = R((m,i),(u,i),(u,g))$ | $R(\theta') = R((m,i),(u,i),(\mathbf{m},g))$ |
|---|---|
| $A$ $A$ $A$ | $A$ $A$ $A$ |
| $C$ $C$ $C$ | $C$ $C$ $C$ |

*Table 7*: Induced agents' preferences given the specified type changes from $\widetilde{\theta}$ to $\theta'$.

In Tables 6 and 7, we have illustrated the idea of partial knitness for two given type profiles. We now show that any relevant pair of type profiles are connected through two appropriate sequences.

**Proposition 7** *The environment $(\Theta, \mathfrak{R})$ in Example 3 is partially knit.*

**Proof of Proposition 7.** Take two pairs $(A,\theta)$, $(C,\widetilde{\theta}) \in A \times \Theta$ such that $\overline{C}(\theta,C,A) = C(\theta,C,A) \neq \varnothing$, $\#C(\theta,C,A) \geq 2$, and for $j \in N\backslash C(\theta,C,A)$, $\widetilde{\theta}_j = \theta_j$. By definition, for all $j \in N$, $\theta_j = (b_j, s_j)$ and $\widetilde{\theta}_j = (\widetilde{b}_j, \widetilde{s}_j)$. We have to show that there exist $\theta' \in \Theta$ and sequences of types $S$ and $\widetilde{S}$ such that $\theta$ leads to $\theta'$ through $S$, $\widetilde{\theta}$ leads to $\theta'$ through $\widetilde{S}$, and the passages from $\theta$ and $\widetilde{\theta}$ to $\theta'$ are, respectively, $A$ and $C$-satisfactory.
Let $\theta' \in \Theta$ be such that $\theta'_j = (b_j, g)$ for any $j \in C(\theta,C,A)$ and $\theta'_j = \theta_j$ for any $j \in N\backslash C(\theta,C,A)$. Define the sequence $S = \{(b_k, g)\}$, where $k \in C(\theta,C,A)$ and $s_k = i$. Note that $I(S)$ is either a singleton or empty. If the latter, let $\theta'$ be $\theta$.
By definition of the preference function in the example, if some agent $j$ prefers $C$ to $A$, the signal profile must be such that at most one agent $k$ has signal $i$: $s_k = i$. Thus, $S$ is well-defined. Moreover, $b_k = m$ since for unswerving jurors to have $C$ over $A$ their signal must be $g$. And by definition of $m$ increasing the support for $g$ implies that preferences remain $C$ over $A$ for agent $k$ (i.e. and $A$-reshuffling) and will be $C$ over $A$ for the other agents.
Therefore, we have defined $S$ to go from $\theta$ to $\theta'$ through $S$ and the passage is $A$-satisfactory.
We now go from $\widetilde{\theta}$ to $\theta'$ by successively changing the type of the agents in $C(\theta,C,A)$, one by one in any order, from to $\widetilde{s}_j \neq g$ to $g$. This set of agents are those in $I(\widetilde{S})$.
By definition of the preference function, if one agent changes her signal by increasing the support for a guilty verdict, then each agents' induced preferences remain either the same as before or change in favor of $C$. Thus, in each one of the above changes, the induced preferences of the agent changing her type is a $C$-monotonic transform of her previous ones (sometimes a $C$-reshuffling).
Now, take any two pairs $(C,\theta)$, $(A,\widetilde{\theta}) \in A \times \Theta$ such that $\overline{C}(\theta,A,C) = C(\theta,A,C) \neq \varnothing$, $\#C(\theta,A,C) \geq 2$, and for $j \in N\backslash C(\theta,A,C)$, $\widetilde{\theta}_j = \theta_j$, a similar argument would work but defining $\theta' \in \Theta$ to be such that $\theta'_j = (b_j, i)$ for any $j \in C(\theta,A,C)$ and $\theta'_j = \theta_j$ for any $j \in N\backslash C(\theta,A,C)$. Define the sequence $S = \{(b_k, i)\}$, where $k \in C(\theta,A,C)$ and $s_k = g$. Note that $I(S)$ is either a singleton or empty. If the latter, let $\theta'$ be $\theta$.
Again, by definition of the preference function in the example, if some agent $j$ prefers $A$ to $C$, the signal profile must be such that only one single agent, or at most two, have signal $g$. In the latter case, none of the two are agent $j$, and both have preferences $C$ over $A$. Thus, $S$ is well-defined. Moreover, by definition of $m$ and $u$ increasing if the single agent with signal $g$ says $i$, that preferences of this agent and those of all other agents will be $A$ over $C$.
Therefore, we have defined $S$ to go from $\theta$ to $\theta'$ through $S$ and the passage is $A$-satisfactory.
We now sequentially go from $\widetilde{\theta}$ to $\theta'$ by successively changing the type of the agents in $C(\theta,A,C)$, one by one in any order, from to $\widetilde{s}_j \neq i$ to $i$. This set of agents are those in $I(\widetilde{S})$.
By definition of agents' preference function, if one agent changes her signal by increasing the support for verdict of innocence, then each agents' induced preferences remain either the same as before or change in favor of $A$. Thus, in each one of the above changes, the induced preferences of the agent changing her type is a $A$-monotonic transform of her previous ones (sometimes a $A$-reshuffling). ■

**Remark 2** *A voting by quota q mechanism is non-constant, ex post incentive compatible, and respectful in the environment in Example 3.*

**Proof of Remark 2.** In Table 8 below we describe all possible results of voting by quota

for different values of $q$ in Example 3. We have four matrices, one for each type of agent 3. In the rows of each matrix we write the four types of agent 1 and in the columns the four types of agent 2. In each cell, we write each agent's best alternative according to their preferences at a given type profile, followed by the outcome of a quota mechanism. When two outcomes appear in a cell, the one in the left stands for the outcome of voting by quota 3 and the right one is the outcome for both quota 1 and 2, which in this example are always the same.
Given Table 8, it is easy to check that these rules are ex post incentive compatible. In addition, they also satisfy anonymity. Note that respectfulness is trivially satisfied in these environments where preferences are strict and alternatives have no private component.

| $\theta_3 = (m,i)$ | $\theta_2 = (m,i)$ | $\theta_2 = (m,g)$ | $\theta_2 = (u,i)$ | $\theta_2 = (u,g)$ |
|---|---|---|---|---|
| $\theta_1 = (m,i)$ | AAA A | AAA A | AAA A | AAA A |
| $\theta_1 = (m,g)$ | AAA A | CCC C | AAA A | CCC C |
| $\theta_1 = (u,i)$ | AAA A | AAA A | AAA A | AAA A |
| $\theta_1 = (u,g)$ | AAA A | CCC C | AAA A | CCC C |
| $\theta_3 = (u,i)$ | $\theta_2 = (m,i)$ | $\theta_2 = (m,g)$ | $\theta_2 = (u,i)$ | $\theta_2 = (u,g)$ |
| $\theta_1 = (m,i)$ | AAA A | AAA A | AAA A | AAA A |
| $\theta_1 = (m,g)$ | AAA A | CCA A/C | AAA A | CCA A/C |
| $\theta_1 = (u,i)$ | AAA A | AAA A | AAA A | AAA A |
| $\theta_1 = (u,g)$ | AAA A | CCA A/C | AAA A | CCA A/C |

| $\theta_3 = (m,g)$ | $\theta_2 = (m,i)$ | $\theta_2 = (m,g)$ | $\theta_2 = (u,i)$ | $\theta_2 = (u,g)$ |
|---|---|---|---|---|
| $\theta_1 = (m,i)$ | AAA A | CCC C | AAA A | CCC C |
| $\theta_1 = (m,g)$ | CCC C | CCC C | CAC A/C | CCC C |
| $\theta_1 = (u,i)$ | AAA A | ACC A/C | AAA A | ACC A/C |
| $\theta_1 = (u,g)$ | CCC C | CCC C | CAC A/C | CCC C |
| $\theta_3 = (u,g))$ | $\theta_2 = (m,i)$ | $\theta_2 = (m,g)$ | $\theta_2 = (u,i)$ | $\theta_2 = (u,g)$ |
| $\theta_1 = (m,i)$ | AAA A | CCC C | AAA A | CCC C |
| $\theta_1 = (m,g)$ | CCC C | CCC C | CAC A/C | CCC C |
| $\theta_1 = (u,i)$ | AAA A | ACC A/C | AAA A | ACC A/C |
| $\theta_1 = (u,g)$ | CCC C | CCC C | CAC A/C | CCC C |

*Table 8.* Each agent's best alternative and outcomes of all voting by quota mechanisms. ■

## Mechanisms with binary outcome problems with a continuous type space

**Proposition 8** *The environment $(\Theta, \mathfrak{R})$ in Example 4 is knit.*

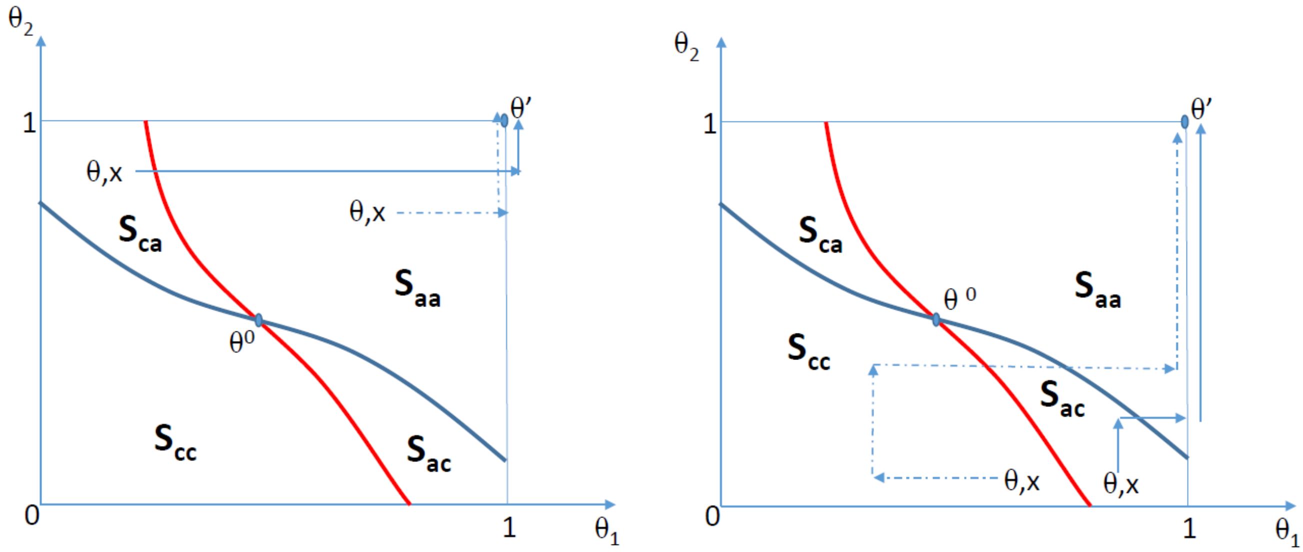
**Proof of Proposition 8.** Given any two pairs $(x,\theta)$, $(z,\widetilde{\theta}) \in A \times \Theta$ we will show that there exist $\theta'$, $S$, $\widetilde{S}$ such that $\theta$ leads to $\theta'$ through $S$, $\widetilde{\theta}$ leads to $\theta'$ through $\widetilde{S}$ and the passages are $x$ and $z$-satisfactory. We choose $\theta' = (1,1)$ independently of the two chosen pairs $(x,\theta)$,

$(z, \widetilde{\theta}) \in A \times \Theta$. In defining the sequence $S$ from $\theta$ to $\theta'$ with $x$ as reference alternative, we distinguish two cases where we will end up analyzing all possible $\theta \in \Theta$. In particular, we cover the case where $\theta$ and $\widetilde{\theta}$ are the same.
<u>Case 1</u>. $\theta \in S_{ca} \cup S_{aa} \cup S^0$. First change the type of agent 1 from $\theta_1 \neq 1$ to 1. Since the function $g_1$ is strictly increasing in type 1, the preferences of agent 1 induced by this change are either an $x$-reshuffling or an $x$-monotonic transform of her original ones. Then change the type of agent 2 from $\theta_2$ to 1. Again, since the function $g_2$ is strictly increasing in type 2, the preferences of agent 2 induced by this change are an $x$-reshuffling of her original ones (see Picture 2.a in Figure 2).
<u>Case 2</u>. $\theta \in S_{ac} \cup S_{cc}$. In this case we may not be able to change types of agents from $\theta_i \neq 1$ to $(1, 1)$ as directly as above.
*Case 2.1.* $\theta$ is a type profile from which we can reach another one in $S_{aa}$ by letting the type of the first agent to be 1. Thus, $S_{aa}$ is not empty. We use the same argument as in Case 1: first change the type of agent 1 from $\theta_1 \neq 1$ to 1. The preferences of agent 1 induced by this change are either an $x$-reshuffling or an $x$-monotonic transform of her original ones. Then change the type of agent 2 from $\theta_2$ to 1. The preferences of agent 2 induced by this change are an $x$-reshuffling of her original ones.
*Case 2.2.* $\theta$ is a type profile from which we can not reach another one in $S_{aa}$ by letting the type of the first agent to be 1. If $S_{aa}$ is not empty, the sequence $S$ must start by previous changes of signals, at most one for each agent, as shown in Picture 2.b in Figure 2, that keep us within the element of the partition where $\theta$ belongs to. The induced preferences resulting from these previous type changes remain unchanged. Then, apply Case 2.1 when reaching a type profile satisfying its statement. If $S_{aa}$ is empty, then either $S^0$ is empty or $S^0 = (1, 1)$. First change the type of agent 2 from $\theta_2 \neq 1$ to 1. The preferences of agent 2 induced by this change are an $x$-reshuffling. Then change the type of agent 1 from $\theta_1$ to 1. The preferences of agent 1 induced by this change are an $x$-reshuffling of her original preferences if $S^0$ is empty or an $x$-monotonic transform if $S^0 = \{(1, 1)\}$.

To define the sequence $\widetilde{S}$ from $\widetilde{\theta}$ to $\theta'$ with $z$ as reference alternative, we would follow a parallel construction to Cases 1 and 2 above. The relevant cases would now be Case 3: $\widetilde{\theta} \in S_{ac} \cup S_{aa} \cup S^0$ and Case 4: $\widetilde{\theta} \in S_{ca} \cup S_{cc}$ where we would consider all possible type profiles $\widetilde{\theta} \in \Theta$ including $\theta$. The proof for the existence of the sequence $\widetilde{S}$ would require a similar argument to those of Cases 1 and 2, respectively, but exchanging the role of agents. See the graphical representation in Figure 3.

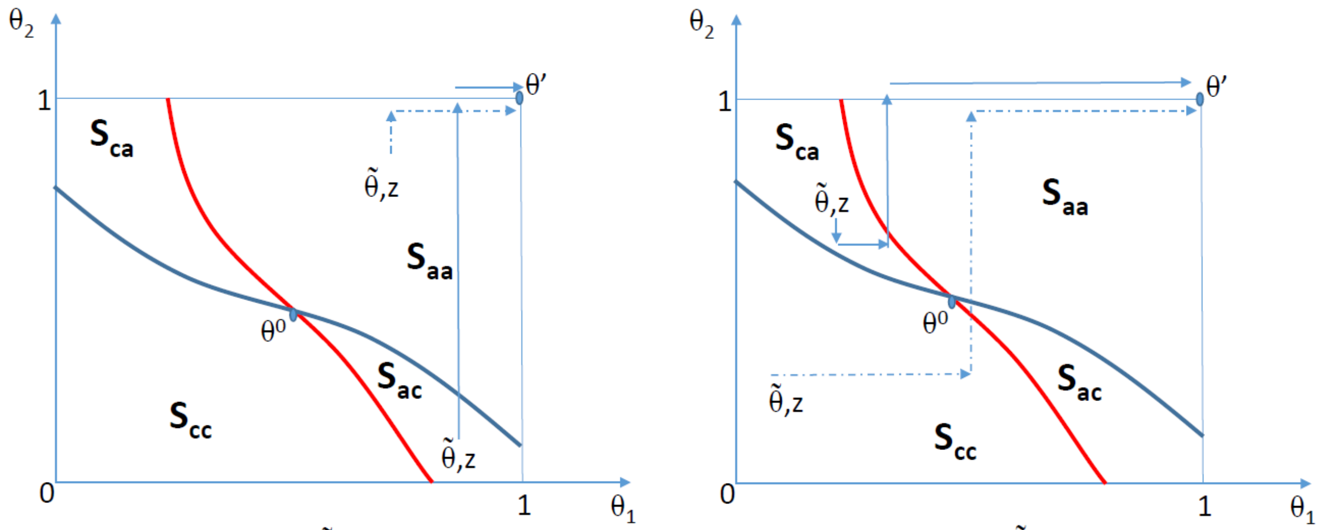
The construction of these passages proves that our environment is knit as we wanted to show. ■

Before engaging in the proof that the environment in Example 5 is partially knit, observe that the changes in the functions $g_i$ imply that the sets $\overline{S}_{ca} = \{\theta \in \Theta : zP_1x \text{ and } zP_2x\}$ and $\overline{S}_{ac} = \{\theta \in \Theta : xP_1z \text{ and } xP_2z\}$ are empty, and that $S^0$ is not a singleton. Due to the specific form of $g_i$ the indifference set is $L$-shaped and thick, as shown in Figure 4.




Picture 2.a. Case 1: $\theta \in S_{ca} \cup S_{aa} \cup s^0$ Picture 2.b. Case 2: $\theta \in S_{ac} \cup S_{cc}$

Figure 2. Changes of agents' types in Cases 1 and 2, proof of Proposition 8.




Picture 3.a. Case 3: $\tilde{\theta} \in S_{ac} \cup S_{aa} \cup s^0$ Picture 3.b. Case 4: $\tilde{\theta} \in S_{ca} \cup S_{cc}$

Figure 3. Changes of agents' types in Cases 3 and 4, proof of Proposition 8.

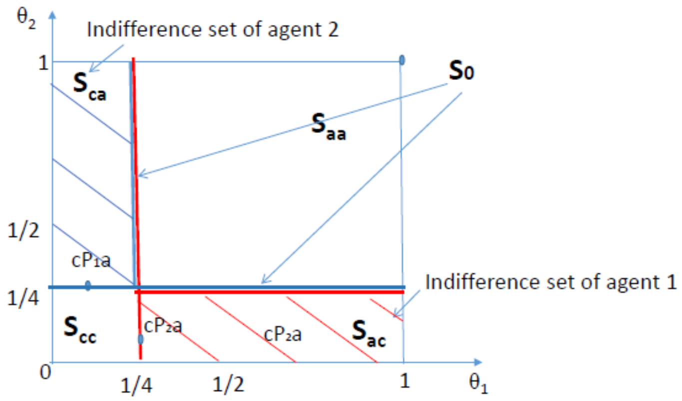



Figure 4. Partition of S in Example 5.

**Proposition 9** *The environment $(\Theta, \mathfrak{R})$ in Example 5 is partially knit.*

**Proof of Proposition 9.** Remember that type profiles are signal profiles. Thus, we identify $s$ with $\theta$. Take any two pairs $(x, \theta)$, $(z, \widetilde{\theta}) \in A \times \Theta$ such that $\overline{C}(\theta, z, x) \neq \varnothing$ and $\#C(\theta, z, x) \geq 2$. These two conditions on $\theta$ imply that we must only consider $\theta \in S_{ca}$, i.e. where agent 1 strictly prefers $z$ to $x$ and agent 2 is indifferent between $x$ and $z$. Define $\theta' = \widetilde{\theta}$.

We have to define $S$ such that $\theta$ leads to $\theta' = \widetilde{\theta}$ through $S$ and the passage is $x$-satisfactory. We distinguish two cases. See the graphical representation of both cases in Figure 5.

Case 1. $\widetilde{\theta} \in S_{aa} \cup S_{ca}$. Define $S = \{\widetilde{\theta}_1, \widetilde{\theta}_2\}$ and $I(S) = \{1, 2\}$. Note that if $\widetilde{\theta}, \theta \in S_{ca}$ the proof is obvious since we move along the same set $S_{ca}$ and no agent preferences change.

Suppose that $\widetilde{\theta} \in S_{aa}$. We first increase the signal of agent 1 to $\theta'_1 = \widetilde{\theta}_1$. The induced preferences of agent 1 are an $x$-monotonic transform of her previous ones. Agent 2 turns to strictly prefer $z$ to $x$, that is, $zR_2(\theta'_1, \theta_2)x$. Decrease or increase now agent 2's signal to $\theta'_2 = \widetilde{\theta}_2$. Note that agent 2's induced preferences are identical to her previous ones, thus, are obviously an $x$-reshuffling of them. So we have gone from $\theta$ to $\theta'$ through adequate types changes with respect to $x$.

Case 2. $\widetilde{\theta} \in S_{cc} \cup S_{ac}$. Define $S = \{\widetilde{\theta}_2, \widetilde{\theta}_1\}$ and $I(S) = \{2, 1\}$. We first decrease the signal of agent 2 to $\theta'_2 = \widetilde{\theta}_2$. The induced preferences of agent 2 are an $x$-monotonic transform of her previous ones $R_2(\theta)$ (since $zP_2(\theta)x$ while $xP_2(\theta_1, \theta'_2)z$). Agent 1 turns to have the same preferences as before, that is, $zR_1(\theta_1, \theta'_2)x$. Now, we decrease or increase agent 1's signal to $\theta'_1 = \widetilde{\theta}_1$. Note that agent 1's induced preferences are either identical to her previous ones (thus, obviously an $x$-reshuffling of those) or an $x$-monotonic transform of $R_1(\theta_1, \theta'_2)$ (since $zP_1(\theta_1, \theta'_2)x$ while $zI_1(\theta')x$). So, we have gone from $\theta$ to $\theta'$ through adequate changes of types with reference $x$.

It remains to consider any two pairs where $(z, \theta)$, $(x, \widetilde{\theta}) \in A \times \Theta$ are such that $\overline{C}(\theta, x, z) \neq \varnothing$ and $\#C(\theta, x, z) \geq 2$, for which a symmetric and similar argument would work.

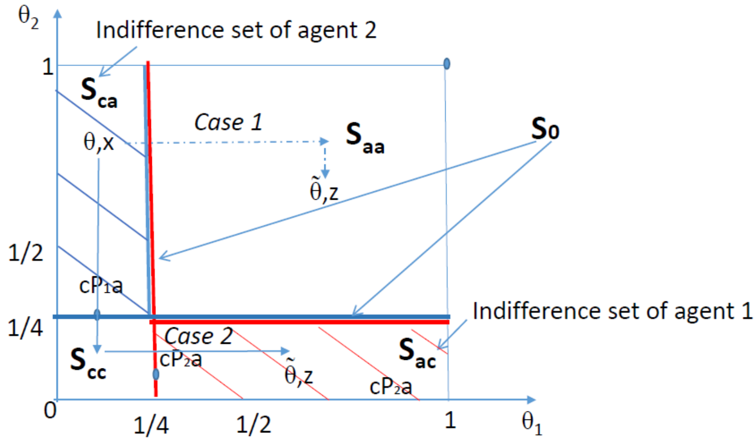



Figure 5. Changes of agents' types, proof of Proposition 9.

■

**Remark 3** *$f_{veto,x}$ is non-constant, ex post incentive compatible, and respectful in the environment $(\Theta, R)$ in Example 5.*

**Proof of Remark 3.** Observe that, by definition, $f_{veto\ x}$ is non-constant and no agent can gain by changing her individual types, since she will either obtain the same or an indifferent one, when deviating, or else obtain her best outcome through by being truthful. Ex post group incentive compatibility is straightforward since changing both types it is impossible to weakly improve both agents, and at least on of them strictly: note that either agent 1 or 2 strictly lose (we need to check 6 cases: $\theta \in S_{aa}$ and $\theta' \in S_{ca}$ or vice versa; $\theta \in S_{ac}$ and $\theta' \in S_{ca}$ or vice versa; and $\theta \in S_{cc}$ and $\theta' \in S_{ac}$ or vice versa). To show that $f_{veto\ x}$ is respectful, note that the only way for agent 1 to remain indifferent according to her initial preferences $R_1(\theta)$ and get a different outcome when changing her type is when $\theta \in S_{ac}$ and $\theta_1' < \frac{1}{4}$ such that $(\theta_1', \theta_2) \in S_{cc}$. However, $R_1(\theta_1', \theta_2)$ is not an $x = f_{veto\ x}(\theta)$-monotonic transform of $R_1(\theta)$. Similarly, for agent 2, to remain indifferent and get a different outcome when changing her type $\theta \in S^0$ and $\theta_2 \geq \frac{1}{4}$, $\theta_2' < \frac{1}{4}$. However, $R_2(\theta_1, \theta_2')$ is not a $z = f_{veto\ x}(\theta)$-monotonic transform of $R_2(\theta)$. ■

# Appendix B. Illustrative examples

In this Appendix we first illustrate the concepts of satisfactoriness and knitness using Example 1 in Remarks 4 and Remark 5. Then, we use Example 6 to show that partial knitness is not necessary for the result in Theorem 4 to hold. Finally, we use Example 7 to illustrate that the necessary conditions in Theorem 3 are not sufficient to escape the constancy conclusions of Theorem 1.

**Remark 4** *The first passage defined in Example 1 is a-satisfactory. The second is not.*

**Proof of Remark 4.** Let $x = a$, $\theta = (\underline{\theta}_1, \underline{\theta}_2)$, $\theta' = (\overline{\theta}_1, \underline{\theta}_2)$, and $S = \{\overline{\theta}_2, \overline{\theta}_1, \underline{\theta}_2\}$ be a sequence of individual types. Note that, $I(S) = \{2, 1, 2\}$ and $t_S = 3$. The passage from $\theta$ to $\theta'$ through $S$ is $a$-satisfactory. To show it, we have to check that for each $h \in \{1, 2, t_S = 3\}$,

$R^{h}_{i(S,h)}(\theta, S)$ is an $a$-monotonic transform of $R^{h-1}_{i(S,h)}(\theta, S)$.
For that, observe first that $R^{0}_{i(S,1)}(\theta, S) = R_2(\underline{\theta}_1, \underline{\theta}_2)$, $R^{1}_{i(S,1)}(\theta, S) = R_2(\underline{\theta}_1, \overline{\theta}_2)$, $R^{1}_{i(S,2)}(\theta, S) = R_1(\underline{\theta}_1, \overline{\theta}_2)$, $R^{2}_{i(S,2)}(\theta, S) = R_1(\overline{\theta}_1, \overline{\theta}_2)$, $R^{2}_{i(S,3)}(\theta, S) = R_2(\overline{\theta}_1, \overline{\theta}_2)$, and $R^{3}_{i(S,3)}(\theta, S) = R_2(\overline{\theta}_1, \underline{\theta}_2)$.
Then, using the table in Example 1, note that the following three facts hold: $R_2(\underline{\theta}_1, \overline{\theta}_2) = a(bc)$ is an $a$-monotonic transform of $R_2(\underline{\theta}_1, \underline{\theta}_2) = b(ac)$ since $U(R_2(\underline{\theta}_1, \overline{\theta}_2), a) = \{a\} \subseteq U(R_2(\underline{\theta}_1, \underline{\theta}_2), a) = \{a, b, c\}$ and $\overline{U}(R_2(\underline{\theta}_1, \overline{\theta}_2), a) = \varnothing \subseteq \overline{U}(R_2(\underline{\theta}_1, \underline{\theta}_2), a) = \{b\}$.
Moreover, $R_1(\overline{\theta}_1, \overline{\theta}_2) = c(ab)$ is an $a$-monotonic transform of $R_1(\underline{\theta}_1, \overline{\theta}_2) = bca$ since $U(R_1(\overline{\theta}_1, \overline{\theta}_2), a) = \{a, b, c\} \subseteq U(R_1(\underline{\theta}_1, \overline{\theta}_2), a) = \{a, b, c\}$ and $\overline{U}(R_1(\overline{\theta}_1, \overline{\theta}_2), a) = \{c\} \subseteq \overline{U}(R_1(\underline{\theta}_1, \overline{\theta}_2), a) = \{b, c\}$.
Finally, $R_2(\overline{\theta}_1, \underline{\theta}_2) = c(ab)$ is an $a$-reshuffling of $R_2(\overline{\theta}_1, \overline{\theta}_2) = c(ab)$ since both preferences coincide.
Now, let $x = a$, $\theta = (\underline{\theta}_1, \underline{\theta}_2)$, $\theta' = (\overline{\theta}_1, \overline{\theta}_2)$, and $S = \{\overline{\theta}_1, \overline{\theta}_2\}$ be a sequence of individual types. Note that, $I(S) = \{1, 2\}$ and $t_S = 2$. The passage from $\theta$ to $\theta'$ through $S$ is not $a$-satisfactory. To show it, observe that for $h = 1$, $R^{h}_{i(S,h)}(\theta, S)$ is not an $a$-monotonic transform of $R^{h-1}_{i(S,h)}(\theta, S)$. By definition, $R^{0}_{i(S,1)}(\theta, S) = R_1(\theta)$ and $R^{1}_{i(S,1)}(\theta, S) = R_1(\overline{\theta}_1, \underline{\theta}_2)$.
Moreover, $R_1(\overline{\theta}_1, \underline{\theta}_2) = c(ab)$ is not an $a$-monotonic transform of $R_1(\theta) = acb$ since $\overline{U}(R_1(\overline{\theta}_1, \underline{\theta}_2), a) = \{a, b, c\} \nsubseteq U(R_1(\theta), a) = \{a\}$ (in fact, $\overline{U}(R_1(\overline{\theta}_1, \underline{\theta}_2), a) = \{c\} \nsubseteq \overline{U}(R_1(\theta), a) = \varnothing$). ■

**Remark 5** *The environment $(\Theta, \mathfrak{R})$ in Example 1 is knit.*

**Proof of Remark 5.** To check that the environment $(\Theta, \mathfrak{R})$ is knit for $\Theta = \{(\underline{\theta}_1, \underline{\theta}_2), (\underline{\theta}_1, \overline{\theta}_2), (\overline{\theta}_1, \underline{\theta}_2), (\overline{\theta}_1, \overline{\theta}_2)\}$, we must prove that all pairs of alternatives and types are pairwise knit, that is, can be connected through satisfactory sequences. To do that, we will show how to choose the appropriate ones for two specific cases, and then argue that all others can be reduced essentially to one of the patterns we shall follow.
<u>Case 1</u>. $(x, \theta) = (a, (\underline{\theta}_1, \underline{\theta}_2))$ and $(z, \widetilde{\theta}) = (b, (\overline{\theta}_1, \underline{\theta}_2))$.
Define $\theta' = \widetilde{\theta} = (\overline{\theta}_1, \underline{\theta}_2)$, $S = \{\overline{\theta}_2, \overline{\theta}_1, \underline{\theta}_2\}$ (thus, $I(S) = \{2, 1, 2\}$ and $t_S = 3$), $\widetilde{S} = \varnothing$ (thus, $I(\widetilde{S}) = \varnothing$ and $t_{\widetilde{S}} = 0$). Note that since $\theta' = \widetilde{\theta}$, then $\widetilde{\theta}$ trivially leads to $\theta'$ through $\widetilde{S}$ and this passage from $\widetilde{\theta}$ to $\theta'$ is $b$-satisfactory. We need to show that $\theta$ leads to $\theta'$ through $S$ and the passage is $a$-satisfactory. For that we need to observe using Table 1 that the three $(t_S)$ following facts hold: $R_2(\underline{\theta}_1, \overline{\theta}_2)$ is an $a$-monotonic transform of $R_2(\underline{\theta}_1, \underline{\theta}_2)$. Moreover, $R_1(\overline{\theta}_1, \overline{\theta}_2)$ is an $a$-monotonic transform of $R_1(\underline{\theta}_1, \overline{\theta}_2)$. Finally, $R_2(\overline{\theta}_1, \underline{\theta}_2)$ is an $a$-reshuffling of $R_2(\overline{\theta}_1, \overline{\theta}_2)$.
<u>Case 2</u>. $(x, \theta) = (c, (\underline{\theta}_1, \underline{\theta}_2))$ and $(z, \widetilde{\theta}) = (a, (\underline{\theta}_1, \overline{\theta}_2))$.
Define $\theta' = (\overline{\theta}_1, \overline{\theta}_2)$, $S = \{\overline{\theta}_1, \overline{\theta}_2\}$ (thus, $I(S) = \{1, 2\}$ and $t_S = 2$), $\widetilde{S} = \{\overline{\theta}_1\}$ (thus, $I(\widetilde{S}) = \{1\}$ and $t_{\widetilde{S}} = 1$). As above, first we need to show that $\theta$ leads to $\theta'$ through $S$ and the passage is $a$-satisfactory. For that we need to observe using Table 1 that the two $(t_S)$ following facts hold: $R_1(\overline{\theta}_1, \underline{\theta}_2)$ is a $c$-monotonic transform of $R_1(\underline{\theta}_1, \underline{\theta}_2)$. Moreover, $R_2(\overline{\theta}_1, \overline{\theta}_2)$ is a $c$-reshuffling of $R_2(\overline{\theta}_1, \underline{\theta}_2)$.
Second, we need to show that $\widetilde{\theta}$ leads to $\theta'$ through $\widetilde{S}$ and the passage is $a$-satisfactory. For that we need to observe using the table that $R_1(\overline{\theta}_1, \overline{\theta}_2)$ is an $a$-monotonic transform of $R_1(\underline{\theta}_1, \overline{\theta}_2)$.
To finish the proof of knitness we should consider all remaining combinations of $(x, \theta)$,

$(z, \widetilde{\theta}) \in A \times \Theta$. Observe that each one of those cases can be embedded in either Case G1 or Case G2 below, which generalize Cases 1 and 2, respectively.
Case G1. $(x, \theta)$ and $(z, \widetilde{\theta})$ such that $x \in \{a, b\}$.
Case G2. $(x, \theta)$ and $(z, \widetilde{\theta})$ such that $x = c$.
To prove knitness for Case G1, consider $\theta' = \widetilde{\theta}$, $\widetilde{S} = \varnothing$, and $S$ will depend on $\theta$ and $\widetilde{\theta}$. Similarly, to prove knitness for Case G2, consider $\theta' = (\overline{\theta}_1, \overline{\theta}_2)$, $S = \{\overline{\theta}_1, \overline{\theta}_2\}$ (thus, $I(S) = \{1, 2\}$ and $t_S = 2$), and $\widetilde{S}$ will depend on $\theta$ and $\widetilde{\theta}$. ■

**Example 6** Consider a private values environment with a finite set of agents $N$, six alternatives $A = \{a_1, a_2, a_3, a_4, y, z\}$, and each agent $i$ has only two strict preferences $\mathcal{R}_i = \{R^2, R^4\}$:

| $R^2$ | $R^4$ |
|---|---|
| $a_2$ | $a_4$ |
| $y$ | $y$ |
| $\mathbf{a}_3$ | $a_1$ |
| $a_4$ | $a_2$ |
| $z$ | $z$ |
| $a_1$ | $\mathbf{a}_3$ |

To show that the environment $\times_{i \in N} \mathcal{R}_i$ is not partially knit, take the two pairs $(a_3, R)$ and $(y, \widetilde{R})$, where $R = (R^2)^n$ and $\widetilde{R} = (R^4)^n$ (note that $C(R, y, a_3) = N$). These two pairs are not pairwise knit since there is no agent's preference $\widehat{R} \neq R^2$ such that $\widehat{R}$ be an $a_3$-monotonic transform of $R^2$ and no agent's preference $\overline{R} \neq R^4$ such that $\overline{R}$ be an $y$-monotonic transform of $R^4$. Thus, we can not construct $R'$.
However, by Theorem 1 in Barberà, Berga, and Moreno (2010) we know that any strategy-proof mechanism on $\times_{i \in N} \mathcal{R}_i$ is strong group strategy-proof since $\times_{i \in N} \mathcal{R}_i$ satisfies sequential inclusion (by their Example 3).

In the following example we illustrate that the condition obtained in Proposition 3, although necessary, is not sufficient for the existence of ex post incentive compatible and respectful mechanisms with range of cardinality $k \geq 2$. To do that we consider a setting with three alternatives and define two environments where there exist $k$ pairs, for $k \in \{2, 3\}$, such that any two pairs are not pairwise knit. For $k = 2$ in the first environment, we show that there is an ex post incentive compatible mechanism and respectful binary mechanism for some pair of alternatives, but in the second environment there is no such binary mechanism. For $k = 3$ in the first environment, there is a full range, ex post incentive compatible, and respectful mechanism, but in the second environment there is no such a mechanism.

**Example 7** Let $N = \{1, 2, 3\}$ and $A = \{x, y, z\}$. Each agent $i$ has two possible types: $\Theta_i = \{\underline{\theta}_i, \overline{\theta}_i\}$. The preference function $\mathfrak{R}$ is defined in Table 9. We write, in each cell, the preferences of the three agents for a given type profile represented by an ordered list from better to worse.

| $\underline{\theta}_3$ | $\underline{\theta}_2$ | | | $\overline{\theta}_2$ | | |
|---|---|---|---|---|---|---|
| $\underline{\theta}_1$ | $R_1(\underline{\theta}_1,\underline{\theta}_2,\underline{\theta}_3)$ xyz | $R_2(\underline{\theta}_1,\underline{\theta}_2,\underline{\theta}_3)$ xyz | $R_3(\underline{\theta}_1,\underline{\theta}_2,\underline{\theta}_3)$ xyz | $R_1(\underline{\theta}_1,\overline{\theta}_2,\underline{\theta}_3)$ yxz | $R_2(\underline{\theta}_1,\overline{\theta}_2,\underline{\theta}_3)$ yzx | $R_3(\underline{\theta}_1,\overline{\theta}_2,\underline{\theta}_3)$ yxz |
| $\overline{\theta}_1$ | $R_1(\overline{\theta}_1,\underline{\theta}_2,\underline{\theta}_3)$ zxy | $R_2(\overline{\theta}_1,\underline{\theta}_2,\underline{\theta}_3)$ yxz | $R_3(\overline{\theta}_1,\underline{\theta}_2,\underline{\theta}_3)$ yxz | $R_1(\overline{\theta}_1,\overline{\theta}_2,\underline{\theta}_3)$ zyx | $R_2(\overline{\theta}_1,\overline{\theta}_2,\underline{\theta}_3)$ zyx | $R_3(\overline{\theta}_1,\overline{\theta}_2,\underline{\theta}_3)$ yxz |

| $\overline{\theta}_3$ | $\underline{\theta}_2$ | | | $\overline{\theta}_2$ | | |
|---|---|---|---|---|---|---|
| $\underline{\theta}_1$ | $R_1(\underline{\theta}_1,\underline{\theta}_2,\overline{\theta}_3)$ yxz | $R_2(\underline{\theta}_1,\underline{\theta}_2,\overline{\theta}_3)$ yxz | $R_3(\underline{\theta}_1,\underline{\theta}_2,\overline{\theta}_3)$ yxz | $R_1(\underline{\theta}_1,\overline{\theta}_2,\overline{\theta}_3)$ yzx | $R_2(\underline{\theta}_1,\overline{\theta}_2,\overline{\theta}_3)$ yxz | $R_3(\underline{\theta}_1,\overline{\theta}_2,\overline{\theta}_3)$ yzx |
| $\overline{\theta}_1$ | $R_1(\overline{\theta}_1,\underline{\theta}_2,\overline{\theta}_3)$ zyx | $R_2(\overline{\theta}_1,\underline{\theta}_2,\overline{\theta}_3)$ xzy | $R_3(\overline{\theta}_1,\underline{\theta}_2,\overline{\theta}_3)$ yzx | $R_1(\overline{\theta}_1,\overline{\theta}_2,\overline{\theta}_3)$ zyx | $R_2(\overline{\theta}_1,\overline{\theta}_2,\overline{\theta}_3)$ zyx | $R_3(\overline{\theta}_1,\overline{\theta}_2,\overline{\theta}_3)$ zyx |

Table 9. Preference function for Example 7

We show that the three pairs $((\overline{\theta}_1,\underline{\theta}_2,\underline{\theta}_3),x)$, $((\underline{\theta}_1,\overline{\theta}_2,\overline{\theta}_3),y)$, $((\overline{\theta}_1,\overline{\theta}_2,\overline{\theta}_3),z)$ are such that any two of them are not pairwise knit. First, observe that there is no $x$-satisfactory passage from $(\overline{\theta}_1,\underline{\theta}_2,\underline{\theta}_3)$ to any other type profile and no $z$satisfactory passage from $(\overline{\theta}_1,\overline{\theta}_2,\overline{\theta}_3)$ to any other type profile. Second, there is no $y$-satisfactory passage from $(\underline{\theta}_1,\overline{\theta}_2,\overline{\theta}_3)$ to neither $(\overline{\theta}_1,\underline{\theta}_2,\underline{\theta}_3)$ nor $(\overline{\theta}_1,\overline{\theta}_2,\overline{\theta}_3)$. Therefore, we can conclude that no two pairs are pairwise knit.

We now define the following mechanism with full range that is ex post incentive compatible.

f:

| $\underline{\theta}_3$ | | | $\overline{\theta}_3$ | | |
|---|---|---|---|---|---|
| | $\underline{\theta}_2$ | $\overline{\theta}_2$ | | $\underline{\theta}_2$ | $\overline{\theta}_2$ |
| $\underline{\theta}_1$ | x | y | $\underline{\theta}_1$ | y | y |
| $\overline{\theta}_1$ | x | z | $\overline{\theta}_1$ | z | z |

We could easily prove by construction that there is no ex post incentive compatible mechanism with range either $\{x,y\}$ or $\{x,z\}$, which is left to the reader. Therefore, by Theorem 2, any two pairs $(\theta,a)$, $(\widetilde{\theta},b)$, $\theta,\widetilde{\theta}\in\Theta$ and $a\neq b$, $a,b$ belonging either to $\{x,y\}$ or $\{x,z\}$ are pairwise knit. However, there is an ex post incentive compatible mechanism with range $\{y,z\}$ where agent 1 is a dictator on its range.

We now modify the preference function by just replacing the preferences of the following agents at two type profiles: $\mathfrak{R}_1(\underline{\theta}_1,\overline{\theta}_2,\underline{\theta}_3)=zyx$, $\mathfrak{R}_1(\overline{\theta}_1,\overline{\theta}_2,\underline{\theta}_3)=zxy$ and $\mathfrak{R}_3(\overline{\theta}_1,\overline{\theta}_2,\underline{\theta}_3)=yzx$.

The three pairs $((\overline{\theta}_1,\underline{\theta}_2,\underline{\theta}_3),x)$, $((\underline{\theta}_1,\overline{\theta}_2,\overline{\theta}_3),y)$, $((\overline{\theta}_1,\overline{\theta}_2,\overline{\theta}_3),z)$ are such that any two of them are not pairwise knit in this modified environment and the proof is identical to the previous one.

In this second environment, there is no ex post incentive compatible mechanism of range neither 3 nor 2. Again, it is left to the reader the impossibility of constructing ex post incentive compatible mechanism with range two.

It is easy to show that there is no full range mechanism satisfying ex post incentive compatible. The proof is by construction: fix any alternative as outcome at type profile $(\overline{\theta}_1,\underline{\theta}_2,\overline{\theta}_3)$, then applying ex post incentive compatibility the mechanism must be constant.

# Appendix C. Single-peakedness

Single-peakedness is undoubtedly the best known property of preference profiles, and one whose use as a domain restriction becomes natural in many applications involving private values.

Our purpose in this appendix is to prove that the set of all strict single-peaked preferences over three alternatives relative to a given order of preferences is partially knit, but not knit. In addition to the intrinsic interest of these facts, we shall discuss why we think that our proof may help to make our admittedly abstract conditions in a better light.

In our reasoning we shall use the fact that when agents' preferences are single-peaked relative to some order of the alternatives, for any triple of alternatives there must be one of them that is never the worst one for any agent. This condition is always necessary for single-peakedness and it is also sufficient for the case of three alternatives. This fact was elaborated upon by Sen and Pattanaik (1969). More recently, Ballester and Haeringer (2011), for the case of more than three alternatives, provide a general characterization involving an additional condition.

We first show that the set of all profiles of single-peaked preferences relative to an order $x < y < z$ is partially knit. To do so, we must prove that for any two pairs $(a, R)$ and $(b, \widetilde{R})$, such that $R$ and $\widetilde{R}$ are in the set of admissible profiles, $bP_ia$ for some agent $i$, and $a$ is different than $b$, there exists some $R'$ which is also admissible and that is *connected* to both $R$ and $\widetilde{R}$ through some $a$-monotonic and $b$-monotonic transforms, respectively. Take any $i$ such that $bP_ia$ and $R_i$ is different to $\widetilde{R}_i$. If $y$ equals $a$ or $b$, choose $R'$ to be such that $bP'_iaP'_ic$. If $y$ is neither $a$ nor $b$, choose $R'$ such that $bP'yP'a$. Since we are in a private values environment, changes in the preferences of each agent do not affect those of others. And notice that the corresponding proposed $R'$ satisfies the monotonicity requirements. That ends the proof.

At the same time, we can also prove directly that the same set is not knit (which we know is true by Proposition 1). It suffices to show that some pair of preferences, $R$ and $\widetilde{R}$ can not be properly connected when associated with alternatives $x$ and $y$, respectively. This is the case when $R$ is such that $xP_iy$ for some $i$, and hence $yP_iz$, because $y$ is never worse. If $\widetilde{R}$ has $y$ as its top alternative, $(x, R)$ and $(y, \widetilde{R})$ cannot be pairwise knit, because the only $x$-monotonic transform of $R$ is $xP'_iyP'_iz$ and it cannot do the job. And this ends our proof.

We hope that these simple arguments can carry to the reader several points that we would like to emphasize. One is that checking for the compliance of our conditions need not always be too complicated. More importantly, that the key difference between the two situations, one of compliance and the other not, is that the comparison between the pairs that we use in showing that single-peakedness is not knit is not required under partial knitness. Thus, in general, what differentiates one condition from the other is that knitness requires connectedness among the profiles in the set under consideration. Finally, let us mention that connectedness, the driving idea behind our conditions, is clearly exemplified by our exercise.