Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xxxxxxx

pp. **X–XX**

FORWARD TRIPLETS AND TOPOLOGICAL ENTROPY ON TREES

Abstract. We provide a new and very simple criterion of positive topological entropy for tree maps. We prove that a tree map f has positive entropy if and only if some iterate f^k has a periodic orbit with three aligned points consecutive in time, that is, a triplet (a, b, c) such that $f^k(a) = b$, $f^k(b) = c$ and b belongs to the interior of the unique interval connecting a and c (a *forward triplet* of f^k). We also prove a new criterion of entropy zero for simplicial n-periodic patterns P based on the non existence of forward triplets of f^k for any $1 \leq k < n$ inside P. Finally, we study the set $\mathcal{X}_n$ of all n-periodic patterns P that have a forward triplet inside P. For any n, we define a pattern that attains the minimum entropy in $\mathcal{X}_n$ and prove that this entropy is the unique real root in $(1, \infty)$ of the polynomial $x^n - 2x - 1$.

Lluís Alsedà

Departament de Matemàtiques
Universitat Autònoma de Barcelona, Cerdanyola del Vallès
Barcelona 08913, Spain

David Juher*

Departament Informàtica, Matemàtica Aplicada i Estadística
Universitat de Girona
c/ Universitat de Girona, 6, Girona 17003, Spain

Francesc Mañosas

Departament de Matemàtiques
Universitat Autònoma de Barcelona, Cerdanyola del Vallès
Barcelona 08913, Spain

(Communicated by the associate editor name)

1. **Introduction and statement of the main results.** This paper deals with discrete dynamical systems defined by the iteration of continuous self-maps on trees. We will give some results relating the positive/zero character of the topological entropy of a map to the combinatorial behavior (or *pattern*) of its periodic orbits. In this section we informally introduce some basic notions and present the main results of the paper.

An *interval* is any space homeomorphic to $[0, 1] \subset \mathbb{R}$. A *tree* is a compact uniquely arcwise connected space which is a union of a finite number of intervals. Any continuous map $f : T \longrightarrow T$ from a tree T into itself will be called a *tree map*. As usual, an *n-periodic orbit* of f is a sequence $\{x_i\}_{i=1}^n \subset T$ such that $f(x_i) = x_{i+1}$

2020 *Mathematics Subject Classification.* Primary: 37E15, 37E25.
Key words and phrases. tree maps, periodic patterns, topological entropy.
Work supported by grants PID2020-118281GB-C31, MTM2017-86795-C3-1-P and 2017 SGR 1617.

* Corresponding author: David Juher.

for $1 \le i < n$ and $f(x_n) = x_1$ (this *time labeling* convention will be used all along this paper for every n-periodic orbit whose elements are indexed from 1 to n).

A classic way of measuring the dynamical richness of a tree map is in terms of its *topological entropy*, first introduced in [1] for continuous maps defined on compact metric spaces. The topological entropy of $f: X \longrightarrow X$, denoted by $h(f)$, is a non-negative real number (or infinity) that measures how the iterates of f mix the points of X. An interval map with positive entropy is *chaotic* in the sense of Li and Yorke [12], and the same is true for general compact metric spaces [8]. On the other hand, the dynamics of a map with zero topological entropy is essentially trivial. It is also well known that the entropy of f is closely related to the number of different periodic orbits exhibited by f and the sizes of such orbits.

In [6], a characterization of positive entropy tree maps was given in terms of the notion of *division.* Informally speaking, a periodic orbit P of a tree map $f: T \longrightarrow T$ is said to have a division if the points of P can be partitioned into subsets that are cyclically permuted by f around a fixed point. One of the main results of [6] (see also [10]) states that a tree map f has positive entropy if and only if there exist $k \ge 1$ and a periodic orbit P of f^k such that P has no division. We will prove that the no division condition can be replaced by a much simpler one. Next we introduce this simple property and state the first main result of this paper.

Given a tree T and a subset $X \subset T$, we define the *connected hull* of X, denoted by $\langle X \rangle_T$ or simply by $\langle X \rangle$, as the smallest closed connected subset of T containing X. When $X = \{x, y\}$ we will write $[x, y]$ to denote $\langle X \rangle$. The notations (x, y), $(x, y]$ and $[x, y)$ will be understood in the natural way. Let $f: T \longrightarrow T$ be a tree map. An ordered set (a, b, c) of three points of T will be called an *aligned triplet* if $b \in (a, c)$. An aligned triplet (a, b, c) will be called a *forward triplet of f* if $f(a) = b$, $f(b) = c$, and $\{a, b, c\}$ is contained in a periodic orbit of f. For instance, (x_8, x_9, x_1) and (x_2, x_3, x_4) are two forward triplets of the map $f: T \longrightarrow T$ shown in Figure 1 (left).

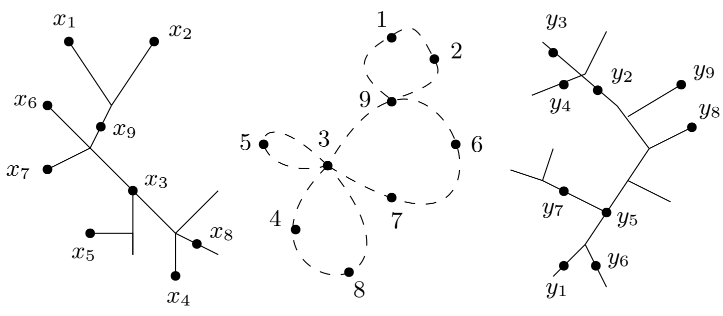
It seems a remarkable fact that the existence of just three consecutive (in time, not necessarily in space) aligned points inside a periodic orbit of a tree map forces positive entropy, as the following theorem states.

Theorem 1.1. *A tree map f has positive topological entropy if and only if there exists $k \ge 1$ such that f^k has a forward triplet.*

In the proof of Theorem 1.1, as well as in the statement of Theorem 1.2 below and all along the paper, the notion of *pattern* of an invariant set will play a central role. Let us introduce this notion.

Given a tree T and a finite subset P of T, the pair (T, P) will be called a *pointed tree.* For a tree map $f: T \longrightarrow T$ having a finite invariant set P, the triplet (T, P, f) will be called a *model.* Given a model (T, P, f), its *pattern* is the equivalence class of all models (S, Q, g) such that, at a combinatorial level, behave like (T, P, f). Here we mean that we do not care about neither the particular topology of the trees nor the action of the maps outside P and Q. Look at Figure 1 for an example of two models of the same pattern. Observe that S and T are not homeomorphic. The central drawing is intended to capture the two ingredients of what we call a *pattern*:

Spatial arrangement of the points. For a pointed tree (T, P), two points x, y of P will be called *consecutive* if $(x, y) \cap P = \emptyset$. Any maximal subset of P consisting only of pairwise consecutive points will be called a *discrete component* of (T, P). We say that two pointed trees (T, P) and (S, Q) are *equivalent* if there exists a bijection $\phi: P \longrightarrow Q$ which preserves discrete components. In the example of Figure 1, just



(T, P, f) (S, Q, g)

FIGURE 1. Two non-homeomorphic trees T and S, with 9-periodic orbits $P = \{x_i\}_{i=1}^9$ and $Q = \{y_i\}_{i=1}^9$ of two respective (unspecified) tree maps $f: T \longrightarrow T$ and $g: S \longrightarrow S$.

FIGURE 2. An interval model (T, P, f) and the corresponding pattern, that can be identified with the permutation $(3, 4, 2, 5, 1)$.

take $\phi: P \longrightarrow Q$ such that $\phi(x_i) = y_{i+2}$ for $1 \le i \le 7$, $\phi(x_8) = y_1$ and $\phi(x_9) = y_2$. Then, ϕ maps the discrete components $\{x_3, x_5\}$, $\{x_3, x_4, x_8\}$, $\{x_3, x_6, x_7, x_9\}$ and $\{x_1, x_2, x_9\}$ of (T, P) respectively into $\{y_5, y_7\}$, $\{y_5, y_6, y_1\}$, $\{y_5, y_8, y_9, y_2\}$ and $\{y_3, y_4, y_2\}$, the discrete components of (S, Q).

Action of the maps. Given two equivalent pointed trees (T, P) and (S, Q), two models (T, P, f) and (S, Q, g) are said to be *equivalent* if $f\big|_P = \phi^{-1} \circ g \circ \phi\big|_P$, where ϕ is a bijection that preserves discrete components. Observe that the map ϕ defined above fulfills this condition for the examples of Figure 1.

A *pattern* is an equivalence class of models by this relation. The central drawing in Figure 1 is the graphic representation of a periodic pattern that will be used all along this paper. Note that there are 9 marks time-labeled from 1 to 9, accounting for the action of each map in the class on the points of the orbit. This points are organized into 4 subsets, $\{3, 5\}$, $\{3, 4, 8\}$, $\{3, 6, 7, 9\}$ and $\{1, 2, 9\}$, the *discrete components* of the pattern.

When we restrict ourselves to the family of continuous self-maps of closed intervals (trees with two endpoints), the points of an n-periodic orbit P are totally ordered and the pattern of P can be clearly identified with a permutation of order n. The notion of interval pattern considered as a permutation has its roots in the well known Sharkovskii's Theorem [16], but it was formalized and developed in the early 1990s [7, 15]. See Figure 2 for an example. On the other hand, the notion of pattern for generic tree maps was first introduced in [2].

In order to capture the minimum dynamical complexity forced by the existence of a given combinatorial behavior, the *entropy* of a pattern $\mathcal{P}$ is defined as

$$h(\mathcal{P}) := \inf\{h(f) : (T, P, f) \text{ is a model of } \mathcal{P}\}.$$

Computing the entropy of a continuous map is a difficult task in general, but the computation of the entropy of a pattern $\mathcal{P}$ can be easily performed thanks to the existence of the so called *canonical models*. Roughly speaking, a *canonical model*

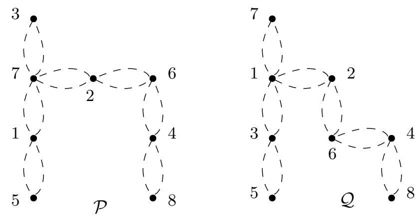


FIGURE 3. Two 8-periodic simplicial patterns $\mathcal{P}$ and $\mathcal{Q}$.

of a pattern $\mathcal{P}$ is an essentially unique and specially simple model (T, P, f) that is determined by the combinatorial data of $\mathcal{P}$ and minimizes the entropy in the set of all models of $\mathcal{P}$. Moreover, the dynamics of f (in particular, its entropy) can be completely described and easily computed using some algebraic tools. We will give the precise definitions in Section 2. The existence of canonical models for patterns on trees has been proved in [2] (the proof is constructive and provides a finite algorithm to build the canonical model from the combinatorial data of the pattern). In the particular case of interval maps, the canonical model of a permutation is nothing but the well known "connect-the-dots" map (see [5] for a list of references).

It is straightforward to define combinatorial versions of the notions of *aligned triplet* and *forward triplet*. Let (T, P, f) be a model. Note that two discrete components of (T, P) are either disjoint or intersect at a single point of P. A point $z \in P$ will be called *inner* if z belongs to at least two discrete components of (T, P). Observe that an ordered subset (a, b, c) of P is an aligned triplet if and only if b is an inner point and $\{a, c\}$ is not contained in a single connected component of $T \setminus \{b\}$. The reader will find easy to convince that this (topological) definition is independent of the particular chosen model of $\mathcal{P}$. So, it makes sense to say that the pattern $\mathcal{P}$ *has an aligned (respectively, forward) triplet* if for any model (T, P, f) of $\mathcal{P}$ there is an aligned (resp. forward) triplet (a, b, c) such that $\{a, b, c\} \subset P$. As an example, the ordered set $(2, 3, 4)$ is a forward triplet of the 9-periodic pattern shown in Figure 1, corresponding to the forward triplets (x_2, x_3, x_4) of $f : T \longrightarrow T$ and (y_4, y_5, y_6) of $g : S \longrightarrow S$.

A pattern $\mathcal{P}$ will be called *simplicial* if each discrete component of $\mathcal{P}$ has two points. See Figure 3 for two examples. Note that, in particular, any interval pattern is simplicial (see Figure 2).

Let $\mathcal{P}$ be an n-periodic pattern. Take a model (T, P, f) of $\mathcal{P}$. For any $1 \leq k < n$, the pattern of (T, P, f^k) will be denoted by $\mathcal{P}^k$. Observe that $\mathcal{P}^k$ is not necessarily periodic. In fact, P consists of the union of $\gcd(n, k)$ periodic orbits of f^k, each one of period $n/\gcd(k, n)$ (see Lemma 2.1.10 of [5]). An n-periodic pattern $\mathcal{P}$ will be called *fully rotational* if $\mathcal{P}^k$ has not forward triplets for all $1 \leq k < n$.

The second main result of this paper provides a new characterization of zero entropy simplicial patterns. It can be viewed as a combinatorial (and stronger) version of Theorem 1.1 for simplicial patterns.

Theorem 1.2. *Let $\mathcal{P}$ be an n-periodic simplicial pattern. Then, $h(\mathcal{P}) = 0$ if and only if $\mathcal{P}$ is fully rotational.*

In the literature, the zero entropy interval patterns are well known (see [5] for a list of classic references). For general tree patterns there are also several criteria of entropy zero [2, 3]. All such criteria require to check that an iterative procedure

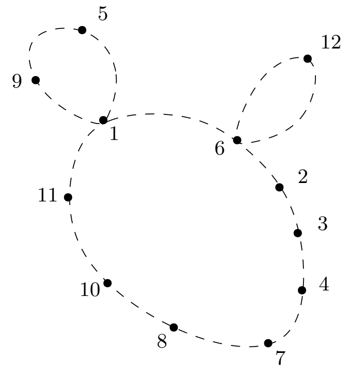


FIGURE 4. A fully rotational 12-periodic pattern $\mathcal{P}$.

of reductions can be carried out leading to a trivial pattern consisting of a single point (see Section 2). The nature of the criterion emanating from Theorem 1.2 is different and requires to check the nonexistence of forward triplets for all relevant iterates of the map over the invariant set. From a practical point of view, this is equivalent to check that for any inner point x of the pattern, x does not separate the points $f^k(x)$ and $f^{-k}(x)$ for every $1 \leq k < n$. As an example, consider the patterns $\mathcal{P}$ and $\mathcal{Q}$ in Figure 3. Note that the inner point 3 of $\mathcal{Q}$ separates 1 and 5. Thus, $(1, 3, 5)$ is a forward triplet of $\mathcal{Q}^2$. Then, $h(\mathcal{Q}) > 0$ by Theorem 1.1. On the other hand, one can check that $\mathcal{P}$ is fully rotational, and thus its entropy is zero by Theorem 1.2.

It is worth noticing that Theorem 1.2 is not true for non-simplicial patterns. As an example, the 12-periodic pattern $\mathcal{P}$ shown in Figure 4 is fully rotational. Indeed, 1 and 6 are the inner points of $\mathcal{P}$, and one can check that for $x = 1, 6$ and any $1 \leq k < 12$, the point x does not separate $f^k(x)$ and $f^{-k}(x)$. But, on the other hand, $h(\mathcal{P}) > 0$ since $\mathcal{P}$ does not satisfy the zero entropy criterion given in [2] (see Section 2).

When a periodic pattern is not fully rotational, then its entropy is positive (just apply Theorem 1.1 to its canonical model (T, P, f)). The third main result of this paper gives in fact a lower bound for its entropy. In [4], the authors considered the problem of determining, for each $n \in \mathbb{N}$, the minimum entropy in the set Pos_n of all n-periodic patterns with positive entropy. Of course every periodic pattern of period 1 or 2 has entropy zero, so the problem makes sense only for $n \geq 3$. Let us introduce the pattern that conjecturally minimizes the entropy in Pos_n.

Let $n \in \mathbb{N}$ with $n \geq 3$. Let $\mathcal{Q}_n$ be the n-periodic pattern whose discrete components are $\{1, n\}$ and $\{1, 2, \ldots, n-1\}$. As an example, in Figure 5 we show $\mathcal{Q}_6$. Observe that $\mathcal{Q}_3$ is the 3-periodic Štefan cycle of the interval [17].

In [4] it is proved that $h(\mathcal{Q}_n) = \log(\lambda_n)$, where λ_n is the unique real root in $(1, \infty)$ of the polynomial $x^n - 2x - 1$. It can be seen that the sequence $(\lambda_n)_{n=1}^{\infty}$ is decreasing and tends to 1 as $n \to \infty$. The main theorem in [4] states that, when n has the particular form $n = p^k$ for a prime p and $k \geq 1$, indeed the pattern $\mathcal{Q}_n$ minimizes the entropy in Pos_n. In consequence, $h(\mathcal{P}) \geq \log(\lambda_n)$ for any pattern $\mathcal{P} \in \mathrm{Pos}_n$. Our third main result states that the same inequality is true for *every* period $n \geq 3$ if we restrict our attention to the non fully rotational patterns.

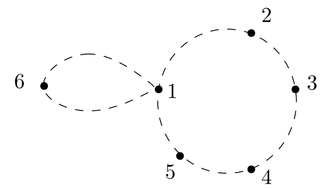


FIGURE 5. The pattern $\mathcal{Q}_6$.

Theorem 1.3. *Let $\mathcal{P}$ be an n-periodic pattern. If $\mathcal{P}$ is not fully rotational, then $h(\mathcal{P}) \geq \log(\lambda_n)$.*

It is worth noticing that the pattern $\mathcal{Q}_n$ is not fully rotational, since has $(n, 1, 2)$ as a forward triplet. Hence, the lower bound of the entropy given in Theorem 1.3 is optimal and is attained by $\mathcal{Q}_n$.

From Theorems 1.2 and 1.3, we immediately get the following corollary.

Theorem 1.4. *Let Sim_n be the set of n-periodic simplicial patterns with positive entropy. Then, $h(\mathcal{P}) \geq \log(\lambda_n)$ for every $\mathcal{P} \in \mathrm{Sim}_n$.*

Presumably, the bound $\log(\lambda_n)$ given by Theorem 1.4 is not optimal in $\mathrm{Sim}_n \subsetneq \mathrm{Pos}_n$, and the problem of determining the minimum entropy simplicial pattern arises. When one restricts to interval patterns, this is a completely solved classic problem. In this case, the minimum positive entropy is attained by the *primary* cycles [9] when n is not a power of 2 and by *extensions* of *minor* cycles [14] when $n = 2^k$.

This paper is organized as follows. In Section 2 we introduce some terminology and notation, and recall some notions and tools that are either common knowledge in the field of Combinatorial Dynamics or recent developments in the particular setting of tree maps. We will use them in Sections 3, 4 and 5 to prove respectively Theorems 1.1, 1.2 and 1.3.

2. **Definitions, terminology and notation.** In this Section we introduce the notion of *canonical model* for tree patterns and explain how to compute the topological entropy of a pattern via the Markov matrix of its canonical model. We also recall the characterization of zero entropy tree patterns first given in [2].

Let T be a tree. For each $x \in T$, we define the *valence* of x to be the number of connected components of $T \setminus \{x\}$. A point of valence different from 2 will be called a *vertex* of T and the set of vertices of T will be denoted by $V(T)$. Each point of valence 1 will be called an *endpoint* of T. The set of such points will be denoted by $\mathrm{En}(T)$. Also, the closure of a connected component of $T \setminus V(T)$ will be called an *edge of* T. Any tree which is a union of $r \geq 2$ intervals whose intersection is a unique point y of valence r will be called an *r-star*, and y will be called its *central point*. Given any subset X of a topological space, we will denote the interior of X by $\mathrm{Int}(X)$. For a finite set P we will denote its cardinality by $|P|$.

The simplest models exhibiting a given pattern are the *monotone* ones, defined as follows. Let $f: T \longrightarrow T$ be a tree map. Given $a, b \in T$ we say that $f\big|_{[a,b]}$ is *monotone* if $f([a, b])$ is either an interval or a point and $f\big|_{[a,b]}$ is monotone as an interval map. Let (T, P, f) be a model. A pair $\{a, b\} \subset P$ will be called a *basic path of* (T, P) if it is contained in a single discrete component of (T, P). We will say that f is *P-monotone* if $\mathrm{En}(T) \subset P$ and $f\big|_{[a,b]}$ is monotone for any basic path $\{a, b\}$.

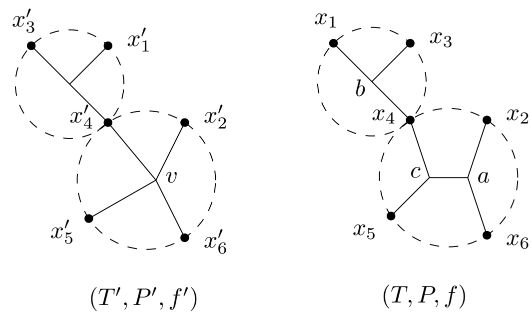


FIGURE 6. Two models of the same pattern.

The model (T, P, f) will then be called *monotone*. In such a case, Proposition 4.2 of [2] states that the set $P \cup V(T)$ is f-invariant. Hence, the map f is also $(P \cup V(T))$-monotone.

Theorem 2.1 (Theorem A of [2])**.** *Let $\mathcal{P}$ be a pattern. Then the following statements hold.*

(a) There exist monotone models of $\mathcal{P}$.
(b) Every monotone model (T, P, f) of $\mathcal{P}$ satisfies $h(f) = h(\mathcal{P})$.

Every pattern has a specially simple monotone model (T, P, f), called *canonical*. It is essentially unique and satisfies the additional property that there are no edges of T whose orbit is either disjoint of P or collapses to a point of P. The existence of the canonical model is stated in Theorem B of [2], whose proof is constructive. Just as an example, the model (T, P, f) in Figure 6(right) is the canonical model of the corresponding pattern. It is not difficult to see that in this case the P-monotonicity of f determines that $f(a) = b$, $f(b) = c$ and $f(c) = c$. Observe also that the model (T', P', f') in Figure 6(left), a representative of the same pattern, cannot be P'-monotone, since in this case we would have $f'(v) \in f'([x_2', x_6']) \cap f'([x_4', x_5']) = [x_3', x_1'] \cap [x_5', x_6'] = \emptyset$.

An n-periodic pattern $\mathcal{P}$ will be called *trivial* if it has only one discrete component. In this case, for $n \geq 2$, let (T, P) be a pointed tree such that T is an n-star with $\mathrm{En}(T) = P = \{x_1, x_2, \ldots, x_n\}$ and let y be its central point. Consider a rigid rotation on T. That is, a model (T, P, f) such that $f(y) = y$ and f maps bijectively $[y, x_i]$ onto $[y, x_{i+1}]$ for $1 \leq i < n$ and $[y, x_n]$ onto $[y, x_1]$. Clearly, (T, P, f) is a monotone model with no invariant forests. Hence, (T, P, f) is the canonical model of $\mathcal{P}$. Therefore, it easily follows that every trivial pattern has entropy 0.

The topological entropy of every map being Q-monotone with respect to a finite invariant set Q containing the vertices of the tree can be easily computed as the logarithm of the spectral radius of the associated *Markov matrix*. Let us recall such standard techniques.

A *combinatorial directed graph* is a pair $\mathcal{G} = (V, U)$ where $V = \{v_1, v_2, \ldots, v_k\}$ is a finite set and $U \subset V \times V$. The elements of V are called the *vertices* of $\mathcal{G}$ and each element (v_i, v_j) in U is called an *arrow* (from v_i to v_j) in $\mathcal{G}$. Such an arrow is usually denoted by $v_i \to v_j$. The notions of *path* and *loop* in $\mathcal{G}$ are defined as usual. The *length* of a path is defined as the number of arrows in the path. The *transition matrix* of $\mathcal{G}$ is a $k \times k$ binary matrix $(m_{ij})_{i,j=1}^k$ such that $m_{ij} = 1$ if and only if there is an arrow from v_i to v_j, and $m_{ij} = 0$ otherwise.

Let (T, Q, f) be a monotone model of a simplicial pattern. That is, $Q \supset V(T)$. Note that, in this case, any connected component of $T \setminus Q$ is an open interval. An interval of T will be called *Q-basic* if it is the closure of a connected component of $T \setminus Q$. Observe that two different Q-basic intervals have pairwise disjoint interiors. Given $K, L \subset T$, we will say that K *f-covers* L if $f(K) \supset L$. Consider a labeling $I_1, I_2, \ldots I_k$ of all Q-basic intervals. The *Markov graph of* (T, Q, f) associated to this labeling is a combinatorial directed graph whose vertices are the Q-basic intervals and there is an arrow from I_i to I_j if and only if I_i f-covers I_j. On the other hand, the *Markov matrix of* (T, Q, f) associated to this labeling is the transition matrix of the corresponding Markov graph of (T, Q, f). Given two different labellings of the set of Q-basic intervals and their associated Markov matrices M and N, there exists a permutation matrix A such that $M = A^T N A$ (where A^T denotes the transpose of A).

For any square matrix M, we will denote its *spectral radius* by $\sigma(M)$. We recall that it is defined as the maximum of the moduli of the eigenvalues of M.

Remark 1. Let (T, Q, f) be a monotone model such that $Q \supset V(T)$. Let M be the Markov matrix of (T, Q, f). By standard arguments (see for instance [5, Theorem 4.4.5]), the topological entropy of f can be computed as

$$h(f) = \log \max\{\sigma(M), 1\}.$$

Recall that if (T, P, f) is the canonical model of a pattern $\mathcal{P}$ then the model $(T, P \cup V(T), f)$ is monotone. Thus, according to the previous paragraphs, we can consider the associated Markov graph and matrix. Since both objects depend only on the canonical model of $\mathcal{P}$, which is uniquely determined by the combinatorial data of the pattern $\mathcal{P}$, they will be respectively called *Markov graph of* $\mathcal{P}$ and *Markov matrix of* $\mathcal{P}$.

Remark 2. Let $\mathcal{P}$ be a pattern and let M be its Markov matrix. From Theorem 2.1(b) and Remark 1 we get that $h(\mathcal{P}) = \log \max\{\sigma(M), 1\}$.

Next we recall the description of zero entropy patterns first given in [2]. Let (T, P, f) be a monotone model of a pattern $\mathcal{P}$. Let π be a basic path of (T, P). We say that $\mathcal{P}$ is *π-reducible* if $f^n(\pi)$ is a basic path of (T, P) for every $n \geq 0$. In this case, let $X = \bigcup_{i \geq 0} \langle f^i(\pi) \rangle$ and let $C_0, C_1, \ldots, C_{p-1}$ be the connected components of X. Note that $P \subset X$. It is easy to see that for each $0 \leq i < p$ there exists j_i such that $f(C_i) \subset C_{j_i}$. Then we take the tree T' obtained from T by collapsing each C_i to a point c_i. Let $\kappa \colon T \longrightarrow T'$ be the standard projection. We set $P' = \kappa(P)$ and define $f' \colon P' \longrightarrow P'$ as $f' = \kappa \circ f \circ \kappa^{-1}$. It is easy to see that $([T', P'], [f'])$ is a well defined pattern, which we call a *π-reduced* (or simply *reduced*) *pattern of* $\mathcal{P}$. The following result (Proposition 9.5 of [4]) summarizes some properties of a reduced pattern for the specific case of periodic patterns.

Proposition 1. *Let $\mathcal{P}$ be an n-periodic pattern that is π-reducible for a basic path π, and let (T, P, f) be a monotone model of $\mathcal{P}$. Let $C_0, C_1, \ldots, C_{s-1}$ be the connected components of $\bigcup_{i \geq 0} \langle f^i(\pi) \rangle$. Then, $n > s \geq 1$ and the following statements hold:*

(a) $\mathrm{En}(C_i) \subset P$ *for* $0 \leq i < s$.
(b) The sets C_i can be labeled in such a way that $f(C_i) = C_{i+1 \bmod s}$ for $0 \leq i < s$. Thus, s divides n, $C_i \cap P$ is an (n/s)-periodic orbit of f^s for each $0 \leq i < s$ and the π-reduced pattern is s-periodic.
(c) The pointed tree $(C_i, C_i \cap P)$ has a unique discrete component for $0 \leq i < s$.

The entropies of a pattern $\mathcal{P}$ and a reduced pattern of $\mathcal{P}$ coincide, as the following result (Proposition 8.1 of [2]) states.

Proposition 2. *Let $\mathcal{P}$ be a pattern. Let $\mathcal{P}'$ be a reduced pattern of $\mathcal{P}$. Then, $h(\mathcal{P}') = h(\mathcal{P})$.*

A pattern will be called *strongly reducible* if there is a finite sequence of reductions leading to a pattern consisting of a single point. The notion of a strongly reducible pattern depends apparently on the chosen sequence of basic paths and monotone models. From the next theorem, which is the characterization of zero entropy patterns given in [2], it follows that this notion is well defined.

Theorem 2.2 (Theorem E of [2])**.** *A pattern has zero entropy if and only if it is strongly reducible.*

3. **Proof of Theorem 1.1.**

The proof of our first main result makes use of several standard concepts about orbits, entropy and patterns of tree maps. In this Section we introduce the notions of *scrambled component*, *division* and *horseshoe*. We also state some basic facts concerning these notions and finally we use them to prove Theorem 1.1.

Let (T, P, f) be a model of a periodic pattern $\mathcal{P}$. Let C be a discrete component of (T, P). We will say that a point $x \in C$ *escapes from C* if $f(x)$ does not belong to the connected component of $T \setminus \{x\}$ that intersects $\operatorname{Int}(\langle C \rangle)$. As an example, observe that the points x_3 and x_9 escape from the discrete component $\{x_3, x_6, x_7, x_9\}$ in the 9-periodic model shown in Figure 1(left). Any discrete component of (T, P) without points escaping from it will be called a *scrambled component* of $\mathcal{P}$. Clearly, these notions do not depend on the particular chosen model of $\mathcal{P}$. So, it makes sense to say that the pattern $\mathcal{P}$ *has a scrambled component*. As an example, the components $\{1, 2, 9\}$ and $\{3, 4, 8\}$ are scrambled for the 9-periodic pattern shown in Figure 1.

Lemma 3.1. *Let $\mathcal{P}$ be a periodic pattern. Then:*

(a) $\mathcal{P}$ has scrambled components.

(b) Let C be a scrambled component of $\mathcal{P}$. For every model (T, P, f) of $\mathcal{P}$, there are fixed points of f in $\operatorname{Int}(\langle C \rangle)_T$.

Statement (a) of Lemma 3.1 is Lemma 4.2 of [4], while (b) is a direct consequence of a more general fact which holds for dendrites, see [11, Theorem 7.2.2(2)].

Next we introduce the notion of a *division* [6]. Let (T, P, f) be a model of an n-periodic pattern $\mathcal{P}$ with $n \geq 2$. Ley y be a fixed point of f such that $y \in \langle P \rangle_T$. Then, $y \notin P$ and there is a discrete component C of (T, P) such that $y \in \operatorname{Int}(\langle C \rangle)$. Let $Z_1, Z_2, \ldots, Z_l$ be the connected components of $T \setminus \operatorname{Int}(\langle C \rangle)$. We will say that the model (T, P, f) (and also the orbit P) *has a p-division with respect to C* (or simply a *p-division*) if there exists $\{M_1, M_2, \ldots, M_p\}$ with $p \geq 2$, a partition of $T \setminus \operatorname{Int}(\langle C \rangle)$, such that each M_i is a union of some of the sets $Z_1, Z_2, \ldots, Z_l$, $f(M_i \cap P) = M_{i+1} \cap P$ for $1 \leq i < p$ and $f(M_p \cap P) = M_1 \cap P$. Again, it is obvious that this definition is independent of the particular chosen model of $\mathcal{P}$, so that it makes sense to say that the pattern $\mathcal{P}$ *has a p-division*.

Several simple facts follow immediately from the definition of division:

Remark 3. Assume that an n-periodic pattern $\mathcal{P}$ has a p-division with respect to a discrete component C. Then, in the notation of the definition of a division,

(a) Each set M_i contains n/p points of P.

(b) If n is prime then $p = n$, $Z_i = M_i$ reduces to one point of P for each $1 \le i \le n$, and C is the only discrete component of $\mathcal{P}$. So, $\mathcal{P}$ is a trivial pattern.
(c) $f(P \cap Z_i) \cap Z_i = \emptyset$ for each $1 \le i \le l$.

The next result states that periodic patterns with a division have only one scrambled component. Its proof, that is left to the reader, follows easily from Remark 3(c).

Lemma 3.2. *If a model (T, P, f) of a periodic pattern has a division with respect to a discrete component C, then C is scrambled and (T, P) has no other scrambled components.*

As it was said in Section 1, it is well known [6, 10] that a tree map f has zero topological entropy if and only if, for every $k \in \mathbb{N}$, each periodic orbit of f^k has a division. This result can be rewritten as follows.

Theorem 3.3. *A tree map f has positive topological entropy if and only if there exist $k \ge 1$ and a periodic orbit P of f^k such that P has no division.*

The following proposition states that having a forward triplet is in fact a particular case of no division.

Proposition 3. *Let (T, P, f) be a model of a periodic pattern $\mathcal{P}$. If (T, P, f) has a forward triplet, then (T, P, f) has no division.*

Proof. If (T, P) has at least two scrambled components, then we are done by virtue of Lemma 3.2. Assume that (T, P) has a unique scrambled component C. Let (a, b, c) be a forward triplet of f. Since $a, b, c \notin \mathrm{Int}(\langle C \rangle)$ and $\langle \{a, b, c\} \rangle$ is an interval, there is a connected component X of of $T \setminus \mathrm{Int}(\langle C \rangle)$ such that either $\{a, b\} \subset X$ or $\{b, c\} \subset X$. Since $f(a) = b$ and $f(b) = c$, condition (c) in Remark 3 does not hold and, in consequence, (T, P, f) has no division. $\square$

The last ingredient to prove Theorem 1.1 is the well known notion of *horseshoe* [13]. Let us recall it. Let $s \ge 2$. For a tree map $f \colon T \longrightarrow T$, an *$s$-horseshoe* is a closed interval $I \subset T$ with $\mathrm{Int}(I) \cap V(T) = \emptyset$ and s closed subintervals $J_1, J_2, \dots, J_s$ of I with pairwise disjoint interiors, such that $f(J_i) = I$ for $1 \le i \le s$.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Assume that $f \colon T \longrightarrow T$ has positive entropy. Then, by Theorem B of [13], there exist sequences $(k_n)_{n=1}^{\infty}$ and $(s_n)_{n=1}^{\infty}$ of positive integers such that for each n the map f^{k_n} has an s_n-horseshoe and

$$\limsup_{n \to \infty} \frac{1}{k_n} \log(s_n) = h(f).$$

Take n such that $s_n \ge 3$. Set $k = k_n$ and $s = s_n$. Since f^k has an s-horseshoe, there is a closed interval $I \subset T$ with $\mathrm{Int}(I) \cap V(T) = \emptyset$ and closed subintervals $J_1, \dots, J_s$ of I with pairwise disjoint interiors such that $f^k(J_i) = I$ for $i = 1, \dots, s$. Take three intervals J, K, L in $\{J_1, \dots, J_s\}$ such that $y \in (x, z)$ for every $x \in J$, $y \in \mathrm{Int}(K)$, $z \in L$. Since J f^k-covers K, K f^k-covers L and L f^k-covers J, by standard covering arguments for continuous maps we get a 3-periodic orbit $P = \{a, b, c\}$ of f^k such that $a \in \mathrm{Int}(J)$, $b = f^k(a) \in \mathrm{Int}(K)$ and $c = f^{2k}(a) \in \mathrm{Int}(L)$. Thus, (a, b, c) is a forward triplet of f^k.

Now assume that there exist a positive integer k and a periodic orbit P of f^k such that P contains a forward triplet of f^k. Then, by Proposition 3, the model (T, P, f^k) has no division. In consequence, $h(f) > 0$ by Theorem 3.3. $\square$

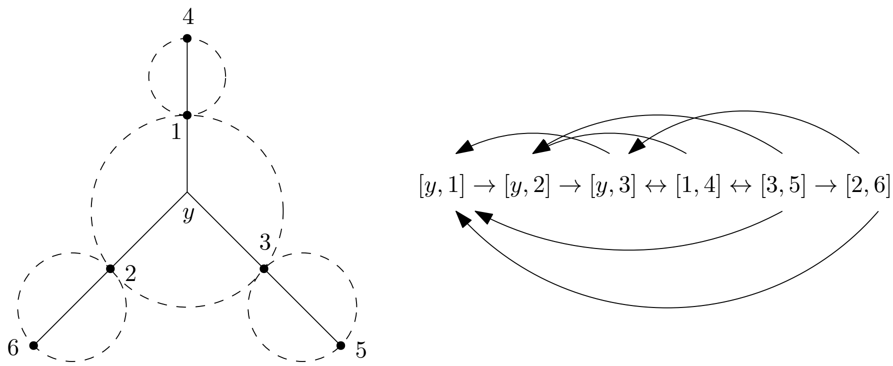


Figure 7. On the left, the canonical model (T, P, f) of a 6-periodic pattern $\mathcal{P}$, for which $f(y) = y$. On the right, the Markov graph of (T, P, f).

A natural question is whether a map f with positive entropy has always a periodic orbit (with high enough period) containing a forward triplet for f. In other words, if we can assume that $k = 1$ in the statement of Theorem 1.1. The canonical model (T, P, f) of the 6-periodic pattern $\mathcal{P}$ shown in Figure 7 is a counterexample. Indeed, one can check that $\mathcal{P}$ is not π-reducible for any basic path π (see Section 2). In consequence, $h(\mathcal{P}) > 0$ by Theorem 2.2. Since (T, P, f) is a canonical model, $h(f) = h(\mathcal{P}) > 0$. On the other hand, the Markov graph of (T, P, f) does not contain paths of the form $I \to J \to K$ such that $I \cup J \cup K$ is contained in an interval of T. Thus, f cannot have periodic orbits with forward triplets.

In view of it, an open question is to find an optimal upper bound for k in the statement of Theorem 1.1, in terms of some combinatorial features of the tree such as the number and/or arrangement of the vertices, edges or endpoints.

4. **Proof of Theorem 1.2.** Before proving Theorem 1.2 we recall that the notion *fully rotational* and the notation $\mathcal{P}^k$ for a pattern $\mathcal{P}$ have been introduced in Section 1. To prove Theorem 1.2 we will state and prove four previous lemmas.

Let $f\colon T \longrightarrow T$ be a tree map and let P be a periodic orbit of f of even period $n = 2k$. For any $x \in P$, the points x and $f^k(x)$ will be called *symmetric* to each other, and $\{x, f^k(x)\}$ will be called a *symmetric pair*. Again, this notion can be directly extended to periodic patterns, so that $\{1, 5\}$ is for instance a symmetric pair of a pattern of period 8, as the ones shown in Figure 3.

Lemma 4.1. *Let $\mathcal{P}$ be a simplicial n-periodic pattern. Then, $\mathcal{P}$ is π-reducible for some basic path π if and only if n is even and any symmetric pair of $\mathcal{P}$ is a discrete component. Moreover, the π-reduced pattern is simplicial and its period is $n/2$.*

Proof. Let (T, P, f) be the canonical model of $\mathcal{P}$. Assume first that $n = 2k$ and that any symmetric pair of $\mathcal{P}$ is a discrete component. Take any $x \in P$ and let π be the symmetric pair $\{x, f^k(x)\}$. Then, $f^i(\pi)$ is a symmetric pair for all $i \geq 0$, and is also a discrete component by hypothesis. In consequence, the pattern $\mathcal{P}$ is π-reducible.

Assume now that $\mathcal{P}$ is π-reducible for a basic path $\pi = \{x, f^k(x)\}$. Set $X = \bigcup_{i\geq 0} \langle f^i(\pi) \rangle$ and let $C_0, C_1, \ldots, C_{s-1}$ be the connected components of X. By Proposition 1(b), they can be labeled in such a way that $f(C_i) = C_{i+1 \bmod s}$ for $0 \leq i < s$.

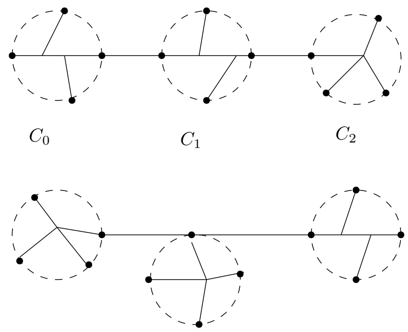


FIGURE 8. The two possible arrangements of the connected components C_0, C_1, C_2 in the proof of Lemma 4.2. The black points belong to P.

Moreover, we can assume that $\pi \subset C_0$. Since $\mathcal{P}$ is simplicial, from (a) and (c) of Proposition 1 we get that every C_i is an interval whose endpoints belong to P and $\mathrm{Int}(C_i) \cap P = \emptyset$. So, $|C_i \cap P| = 2$ for $0 \leq i < s$ and, in consequence, n is even, $s = k = n/2$ and $C_i = [f^i(x), f^{i+k}(x)]$ for $0 \leq i < k$. In other words, every symmetric pair of $\mathcal{P}$ is a discrete component. Finally, to see that the π-reduced pattern is simplicial, recall that it is obtained by collapsing every C_i to a point and observe that, since $\mathcal{P}$ is simplicial, $T \setminus X$ is a union of pairwise disjoint open intervals. □

Note that the characterization of zero entropy patterns given by Theorem 2.2, together with the iterative use of Lemma 4.1, yields that the period of any simplicial periodic pattern with entropy zero is a power of 2, a fact that is well known for the particular case of interval patterns.

Lemma 4.2. *Let $\mathcal{P}$ be a periodic pattern that is π-reducible for a basic path π and let $\mathcal{Q}$ be the π-reduced pattern. If $\mathcal{P}$ is fully rotational, then $\mathcal{Q}$ is fully rotational.*

Proof. Let (T, P, f) and (S, Q, g) be canonical models of $\mathcal{P}$ and $\mathcal{Q}$, respectively. We will prove the lemma by way of contradiction. So, assume that g has a forward triplet (c_0, c_1, c_2) contained in Q. So, $g(c_0) = c_1$, $g(c_1) = c_2$ and $c_1 \in \mathrm{Int}(\langle c_0, c_2 \rangle_S)$. By construction of the π-reduced pattern, there exist three connected components C_0, C_1, C_2 of $\bigcup_{i \geq 0} \langle f^i(\pi) \rangle_T$, that can be respectively identified after collapse with the points c_0, c_1, c_2, such that $f(C_0) = C_1$ and $f(C_1) = C_2$. By Proposition 1(a), $\mathrm{En}(C_i) \subset P$ for $i = 0, 1, 2$. Since a tree is uniquely arcwise connected, there exists at least one point $x \in \mathrm{En}(C_1) \subset P$ such that $x \in \mathrm{Int}(\langle a, b \rangle_T)$ for any pair $a \in C_0$, $b \in C_2$ (in fact, there are at most two points $x \in \mathrm{En}(C_1)$ satisfying this property, see Figure 8). In particular, $x \in \mathrm{Int}(\langle f^{-1}(x), f(x) \rangle_T)$ and, consequently, $(f^{-1}(x), x, f(x))$ is a forward triplet for f on P, a contradiction since $\mathcal{P}$ is fully rotational. □

Next we study some properties of the fully rotational simplicial patterns. To do it, the following simple terminology will be appropriate. Let $\mathcal{P}$ be a fully rotational simplicial pattern. Let (T, P, f) be the canonical model of $\mathcal{P}$. We will say that two points $x, y \in P$ are *adjacent* to each other if $\{x, y\}$ is a discrete component. A discrete component C of $\mathcal{P}$ will be called *extremal* if C contains an endpoint of T. As an example, the components $\{1, 7\}$, $\{3, 5\}$, $\{4, 8\}$ are the three extremal components of the pattern $\mathcal{Q}$ in Figure 3.

The following simple result states that the period of a fully rotational simplicial pattern has to be even. Moreover, the extremal discrete components are symmetric pairs.

Lemma 4.3. *Let $\mathcal{P}$ be a fully rotational n-periodic simplicial pattern. Then, n is even and every extremal discrete component of $\mathcal{P}$ is a symmetric pair.*

Proof. Let (T, P, f) be the canonical model of $\mathcal{P}$. Take any extremal component $\{x, z\}$ of $\mathcal{P}$, with $x \in \mathrm{En}(T)$. Then $z = f^k(x)$ for some $1 \leq k < n$. Since $\mathcal{P}$ is fully rotational, f^k does not have forward triplets. Therefore, $f^k(z) = x$ and $f^{2k}(x) = x$. In consequence, $n = 2k$ and $\{x, z\}$ is a symmetric pair. □

Lemma 4.4. *Let $\mathcal{P}$ be a fully rotational n-periodic simplicial pattern. Then every symmetric pair is a discrete component of $\mathcal{P}$.*

Proof. By Lemma 4.3, n is even. Set $n = 2k$. Let (T, P, f) be the canonical model of $\mathcal{P}$. Fix $e \in P \cap \mathrm{En}(T)$. For any $x \in P \setminus \{e\}$, denote by $\mathcal{B}_x$ the set of all connected components of $T \setminus \{x\}$ that do not contain e. Such connected components will be called *outer components starting at* x. Note that $\mathcal{B}_x = \emptyset$ if and only if $x \in \mathrm{En}(T)$. For any outer component B starting at x, we will say that B is *of even type* if $B \cap P$ is a union of symmetric pairs and every symmetric pair in B is a discrete component. We will say that B is *of odd type* if $B \cap P$ is a (possibly empty) union of symmetric pairs plus the point $f^k(x)$, that is adjacent to x, and every symmetric pair in B is a discrete component. Since every point in P has a unique symmetric point, it follows that

$$\text{for each } x \in P \setminus \{e\}, \text{ there is at most one } B \in \mathcal{B}_x \text{ of odd type} \tag{1}$$

and in this case $|B \cap P|$ is odd, while $|B \cap P|$ is even when B is of even type.

For a point $x \in P \setminus \{e\}$, we label by ($\star$) the following property:

$$\text{given any } B \in \mathcal{B}_x, \text{ either } B \text{ is of even type or } B \text{ is of odd type.} \tag{$\star$}$$

We claim that

$$\text{every } x \in P \setminus \{e\} \text{ satisfies } (\star). \tag{2}$$

Let us see that the lemma follows from this claim. Indeed, let z be the only point of P adjacent to e. By Lemma 4.3, $\{e, z\}$ is a symmetric pair. On the other hand, (1), (2) and the fact that $n - 2$ is even imply that all outer components starting at z are of even type. As a consequence, every symmetric pair is a discrete component and the lemma follows.

To prove (2), we use a kind of *topological induction*. First of all (step 0 of the induction), observe that every endpoint satisfies ($\star$) by definition.

Now consider any $x \in P \setminus \mathrm{En}(T)$. Note that, in the set of all points of P adjacent to x, only one belongs to the interval $[e, x)$. The remaining points will be called *outer neighbours of* x. Observe that every outer component starting at x can be written in terms of the outer components starting at an outer neighbour v of x as

$$(x, v] \bigcup_{B \in \mathcal{B}_v} B.$$

What we will prove to complete the induction process is that, for every $x \in P \setminus \mathrm{En}(T)$,

$$\text{if all outer neighbours of } x \text{ satisfy } (\star), \text{ then } x \text{ satisfies } (\star). \tag{3}$$

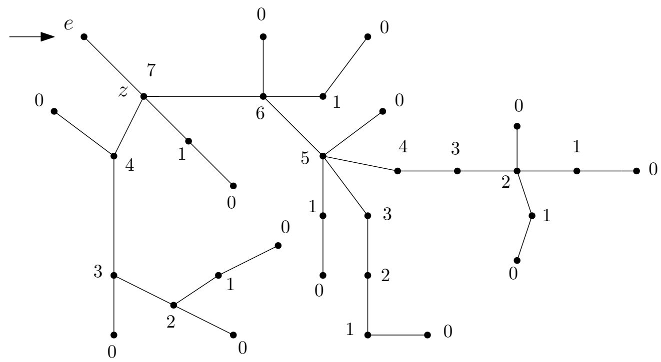


FIGURE 9. Topological induction

It is easy to see that (2) follows using (3) iteratively. Indeed, assign to each point in $P \setminus \{e\}$ an integer label i as follows. If $x \in \mathrm{En}(T)$, set $i = 0$. Otherwise, let i be the maximum of the cardinalities of the sets $P \cap (x, y]$, where y ranges over all endpoints that do not belong to the unique connected component of $T \setminus \{x\}$ containing e. Note that the label of each point is the maximum of the labels of its outer neighbours plus one. As an example, see Figure 9. The label of each point x indicates the induction step in which one can prove, using (3), that x satisfies ($\star$) since all outer neighbours of x satisfy ($\star$).

Collecting it all, to end the proof of the lemma it only remains to show that (3) holds for any $x \in P \setminus \mathrm{En}(T)$. We have to show that all outer components starting at x have type even or odd.

Let us choose one of such outer components, $(x, v] \bigcup_{B \in \mathcal{B}_v} B$, where v is an outer neighbour of x. Recall (1) that at most one $B \in \mathcal{B}_v$ has odd type. Assume first that there is one outer component of odd type starting at v. Then, $(x, v] \bigcup_{B \in \mathcal{B}_v} B$ has even type and we are done. Assume now that all outer components from v are even. In this case, we will show that v is the symmetric point of x. Then, $(x, v] \bigcup_{B \in \mathcal{B}_v} B$ will be of odd type and (3) will be proved.

If $v \in \mathrm{En}(T)$, then $\{v, x\}$ is a symmetric pair by Lemma 4.3 and we are done. Assume now that $v \notin \mathrm{En}(T)$. Set $P_v := \bigcup_{B \in \mathcal{B}_v} (B \cap P)$. To prove that v is the symmetric point of x, for any $a \in \{v\} \cup P_v$ we define the *x-twin* of a as follows: take the unique $r \in \{1, 2, \dots, n-1\}$ such $f^r(a) = x$. Then, the *x-twin of a* is, by definition, $f^r(x)$. Note that if b is the x-twin of a, then a is the x-twin of b, since $f^r(a) = x$ and $f^r(x) = b$ imply that $f^{n-r}(b) = x$ and $f^{n-r}(x) = a$. So, it makes sense to say that $\{a, b\}$ is an *x-twin pair*. Note also that a itself is the x-twin of a if and only if $\{a, x\}$ is a symmetric pair.

Observe that if $a \in \{v\} \cup P_v$ and $f^r(a) = x$, then $(a, x, f^r(x))$ is a forward triplet of f^r if and only if $f^r(x) \notin \{v\} \cup P_v$. So, from the fact that $\mathcal{P}$ is fully rotational we get that the x-twin of every $a \in \{v\} \cup P_v$ (that is different from a unless a is the symmetric point of x) belongs also to $\{v\} \cup P_v$. Since $|\{v\} \cup P_v|$ is odd, it follows that $\{v\} \cup P_v$ is a union of x-twin pairs plus the symmetric point of x. Since we are assuming that every component in $\mathcal{B}_v$ has even type, no component in $\mathcal{B}_v$ can contain the symmetric point of x, and necessarily v, that is adjacent to x, is the symmetric point of x. $\square$

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let (T, P, f) be the canonical model of $\mathcal{P}$, in such a way that $h(f) = h(\mathcal{P})$. To prove the "only if" part of the statement, assume that $\mathcal{P}^k$ has a forward triplet. Then, $h(f) > 0$ by Theorem 1.1.

To prove the "if" part of the statement, assume that $\mathcal{P}$ is fully rotational. That is, for any $1 \leq k < n$, $\mathcal{P}^k$ has not forwards triplets. We have to show that $h(\mathcal{P}) = 0$. By Lemma 4.4, n is even and every symmetric pair is a discrete component. In consequence, by Lemma 4.1, $\mathcal{P}$ is π-reducible for some basic path π, and the π-reduced pattern $\mathcal{Q}$ is simplicial and has period $n/2$. Moreover, $\mathcal{Q}$ is fully rotational by Lemma 4.2. So, we can replace $\mathcal{P}$ by $\mathcal{Q}$ and repeat the argument to get that $\mathcal{Q}$ is also reducible for some basic path. It is clear that going on we obtain a finite sequence of reductions leading to a pattern consisting of a single point. So, $\mathcal{P}$ is strongly reducible and, by Theorem 2.2, its entropy is 0. □

5. **Proof of Theorem 1.3.** We recall that the patterns $\mathcal{Q}_n$ are our candidates for minimum (positive) entropy in the class of n-periodic patterns. They were defined in page 5. Recall also that the entropy of $\mathcal{Q}_n$ is $\log(\lambda_n)$, where λ_n is the unique real root of the polynomial $x^n - 2x - 1$ in $(1, \infty)$. The following result (Proposition 3.1 of [4]) summarizes some properties of the sequence $(\lambda_n)_{n \geq 3}$.

Proposition 4. *Let n be a positive integer with $n \geq 3$. Then:*

(a) $\lambda_{n+1} < \lambda_n$
(b) $(\lambda_n)^{1/k} > \lambda_{kn}$ *for every $k \in \mathbb{N}$ with $k \geq 2$.*

We need to recover also the following result (Theorem 8.1 of [4]).

Theorem 5.1. *Let $\mathcal{P}$ be an n-periodic pattern with two discrete components. If $\mathcal{P}$ has no division, then $h(\mathcal{P}) \geq \log(\lambda_n)$.*

To prove Theorem 1.3 we need a stronger version of the previous result.

Theorem 5.2. *Let $\mathcal{P}$ be an n-periodic pattern with two discrete components. If $h(\mathcal{P}) > 0$, then $h(\mathcal{P}) \geq \log(\lambda_n)$.*

Proof. Let us see that we can restrict to patterns that are not π-reducible for any basic path π. Indeed, let (T, P, f) be the canonical model of $\mathcal{P}$. If $\mathcal{P}$ is π-reducible for a basic path π, then let $C_1, C_2, \dots, C_s$ be the connected components of $X = \bigcup_{i \geq 0} \langle f^i(\pi) \rangle$. By Proposition 1(b), the π-reduced pattern $\mathcal{P}'$ obtained from $\mathcal{P}$ by collapsing every C_i to a point is s-periodic, and its entropy coincides with that of $\mathcal{P}$ by Theorem 2. In particular, $\mathcal{P}'$ cannot have a single discrete component because the entropy of a trivial pattern is zero. In consequence, $\mathcal{P}'$ has two discrete components (Figure 10). Moreover, since $s < n$, $\lambda_n < \lambda_s$ by Proposition 4(b) and the theorem will be proved if we show that $h(\mathcal{P}') \geq \log(\lambda_s)$. It is clear that this process can be iterated, if necessary, to finally obtain a periodic pattern with the same entropy as $\mathcal{P}$ and two discrete components that is not reducible for any basic path.

From the previous paragraph, from now on we will assume that $\mathcal{P}$ is not π-reducible for any basic path π. If $\mathcal{P}$ has no division, then we are done by Theorem 5.1. So, let us assume that $\mathcal{P}$ has a p-division ($p \geq 2$) with respect to a discrete component A, and let B be the other discrete component of $\mathcal{P}$. By the definition of a division, there exists $\{M_1, M_2, \dots, M_p\}$, a partition of $T \setminus \mathrm{Int}(\langle A \rangle)$, such that each M_i is a union of some connected components of $T \setminus \mathrm{Int}(\langle A \rangle)$, $f(M_i \cap P) = M_{i+1} \cap P$ for $1 \leq i < p$ and $f(M_p \cap P) = M_1 \cap P$. Observe that all sets $M_i \cap P$ are contained

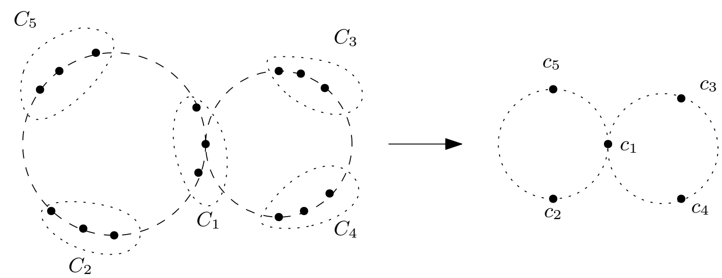


FIGURE 10. A 15-periodic pattern with two discrete components and a π-reduced 5-pattern.

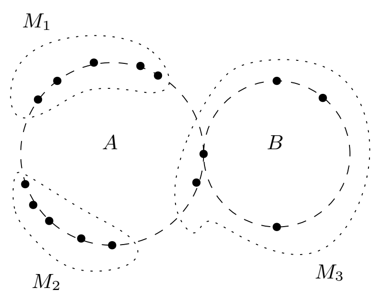


FIGURE 11. A 15-periodic pattern with two discrete components and a 3-division.

in A, except one (we can assume that it is $M_p \cap P$ without loss of generality), that has the form $M_p \cap P = B \cup C$, where C is either empty or a subset of A. See Figure 11. Now let $\mathcal{P}'$ be the pattern of $(T, M_p \cap P, f^p)$, which is (n/p)-periodic. We note that

$$\text{Every basic path of } \mathcal{P}' \text{ is also a basic path of } \mathcal{P}. \tag{4}$$

We claim that $\mathcal{P}'$ is not π-reducible for any basic path π. To prove the claim, assume by way of contradiction that there exists $\pi = \{a, b\}$ such that $\{f^{ip}(a), f^{ip}(b)\}$ is a basic path of $\mathcal{P}'$ for all $i \geq 0$. Since $M_j \subset A$ for $1 \leq j < p$, $\{f^{ip+j}(a), f^{ip+j}(b)\}$ is a basic path of $\mathcal{P}$ for all $i \geq 0$ and $1 \leq j < p$. This fact, together with (4), implies that $\{f^k(a), f^k(b)\}$ is a basic path of $\mathcal{P}$ for all $k \geq 0$ and $\mathcal{P}$ is π-reducible, a contradiction that proves the claim.

Since every trivial pattern is π-reducible, from the previous claim we get that $\mathcal{P}'$ has not one but two connected components. In particular its period, n/p, is at least 3. Moreover,

$$p \cdot h(\mathcal{P}) = p \cdot h(f) = h(f^p) \geq h(\mathcal{P}'). \tag{5}$$

It is clear that if $\mathcal{P}'$ has a division, then the above procedure can be repeated with $\mathcal{P}'$ instead of $\mathcal{P}$. After a finite number of steps and using (5) iteratively, we obtain a sequence of periodic patterns $(\mathcal{P}_i)_{i=1}^k$ with $\mathcal{P}_1 = \mathcal{P}$ such that:

1. Every $\mathcal{P}_i$ has two discrete components and is not reducible for any basic path, in particular $h(\mathcal{P}_i) > 0$
2. $\mathcal{P}_i$ has a p_i-division for $1 \leq i < k$
3. $\mathcal{P}_k$ has no division
4. The period of $\mathcal{P}_i$ is p_i times that of $\mathcal{P}_{i+1}$ for $1 \leq i < k$

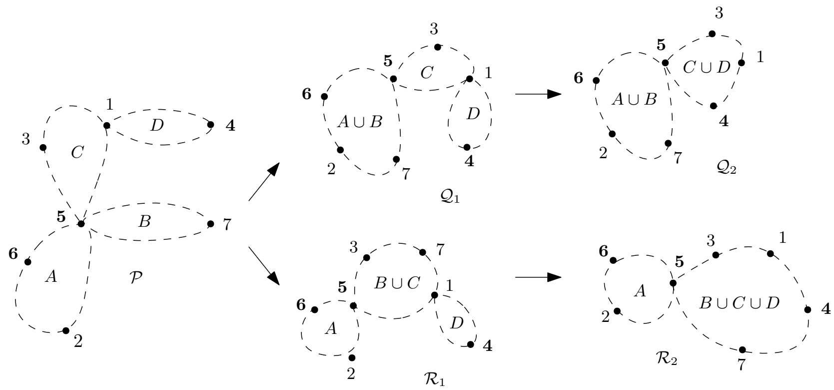


FIGURE 12. Two possible sequences $\mathcal{P} \geq \mathcal{Q}_1 \geq \mathcal{Q}_2$ and $\mathcal{P} \geq \mathcal{R}_1 \geq \mathcal{R}_2$ of pull outs leading to two (different) complete openings of $\mathcal{P}$ with respect to the forward triplet $(4, 5, 6)$.

5. $p_i \cdot h(\mathcal{P}_i) \geq h(\mathcal{P}_{i+1})$ for $1 \leq i < k$.

From (4), the period of $\mathcal{P}_k$ is $m := n/(p_1 p_2 \cdots p_{k-1})$. Using (5) we get that $p_1 p_2 \cdots p_{k-1} h(\mathcal{P}) \geq h(\mathcal{P}_k)$. On the other hand, from (1), (3) and Theorem 5.2 it follows that $h(\mathcal{P}_k) \geq \log(\lambda_m)$. Putting all together and using Proposition 4(b) yields

$$h(\mathcal{P}) \geq \log(\lambda_m)^{\frac{1}{p_1 p_2 \cdots p_{k-1}}} > \log(\lambda_{m p_1 p_2 \cdots p_{k-1}}) = \log(\lambda_n).$$

□

The last ingredient we need to prove Theorem 1.3 is a tool, first introduced in [4], that allows us to compare the entropies of two patterns $\mathcal{P}$ and $\mathcal{Q}$ when $\mathcal{Q}$ has been obtained by joining together several discrete components of $\mathcal{P}$. For the sake of brevity, here we will give a somewhat informal (tough completely clear) version of this procedure.

Let (T, P, f) be a model of a pattern $\mathcal{P}$. Let $x \in P$ be an inner point and let A, B be two discrete components of (T, P) intersecting at x. If we join together A and B to get a new discrete component $A \cup B$ and keep intact the remaining components, we get a new pattern $\mathcal{Q}$. We will say that $\mathcal{Q}$ is a *pull out of* $\mathcal{P}$ *with respect to the inner point* x *and the discrete components* A *and* B, and will write $\mathcal{P} \geq \mathcal{Q}$. As an example, see Figure 12, where $\mathcal{Q}_1$ is a pull out of $\mathcal{P}$ with respect to the inner point 5 and the discrete components $A = \{2, 5, 6\}$ and $B = \{2, 7\}$, while $\mathcal{R}_1$ is a pull out of $\mathcal{P}$ with respect to the inner point 5 and the discrete components B and $C = \{1, 3, 5\}$.

Let $\mathcal{P}$ be a pattern with an aligned triplet (a, b, c). A pattern $\mathcal{Q}$ is said to be a *complete opening of* $\mathcal{P}$ *with respect to* (a, b, c) if the following conditions hold:

1. there exists a sequence of $k \geq 1$ pull outs $\mathcal{P} \geq \mathcal{P}_1 \geq \ldots \geq \mathcal{P}_k = \mathcal{Q}$
2. $\mathcal{Q}$ has two discrete components
3. (a, b, c) keeps being an aligned triplet of $\mathcal{Q}$

In general, there can be several complete openings with respect to an aligned triplet. See the examples of Figure 12, where $\mathcal{Q}_2$ and $\mathcal{R}_2$ are two different complete openings of $\mathcal{P}$ with respect to $(4, 5, 6)$.

As one may expect from intuition, the entropy of a model decreases when performing a pull out, as the following result (Theorem 5.3 of [4]) states.

Theorem 5.3. *Let $\mathcal{P}$ and $\mathcal{Q}$ be n-periodic patterns. If $\mathcal{P} \geq \mathcal{Q}$ then $h(\mathcal{P}) \geq h(\mathcal{Q})$.*

We get an immediate consequence from Theorem 5.3 and property (1) of the definition of a complete opening.

Corollary 1. *Let $\mathcal{Q}$ be a complete opening of an n-periodic model $\mathcal{P}$ with respect to an aligned triplet. Then, $h(\mathcal{P}) \geq h(\mathcal{Q})$.*

Summarizing, a pull out is a mechanism that allows us to reduce both the entropy and the number of discrete components, leading to a very simple pattern with only two discrete components when we perform a complete opening. If the aligned triplet (a, b, c) of the definition of a complete opening is in addition a forward triplet, then we can assure that the entropy of $\mathcal{Q}$ stays positive and then we can use Theorem 5.2. This is the basic idea in the proof of Theorem 1.3.

Proof of Theorem 1.3. Since $\mathcal{P}$ is not fully rotational, there exists a forward triplet (a, b, c) of $\mathcal{P}^k$ for some $1 \leq k < n$. Note that (a, b, c) is an aligned triplet of $\mathcal{P}$. Let $\mathcal{Q}$ be a complete opening of $\mathcal{P}$ with respect to (a, b, c). By Corollary 1, $h(\mathcal{P}) \geq h(\mathcal{Q})$. Thus, to prove the theorem it is enough to see that $h(\mathcal{Q}) \geq \log(\lambda_n)$. Let (S, Q, g) be the canonical model of $\mathcal{Q}$. By definition of a complete opening, $\mathcal{Q}$ has two discrete components and (a, b, c) is an aligned triplet of $\mathcal{Q}$. Equivalently, b is the unique inner point of (S, Q) and separates a and c. Moreover, since (a, b, c) is a forward triplet of $\mathcal{P}^k$, then $g^k(a) = b$ and $g^k(b) = c$. In consequence, (a, b, c) is a forward triplet of g^k. Thus, by Theorem 1.1, the entropy of g, which equals $h(\mathcal{Q})$ because (S, Q, g) is a canonical model, is positive. Since $\mathcal{Q}$ is an n-periodic pattern with two discrete components and positive entropy, $h(\mathcal{Q}) \geq \log(\lambda_n)$ by Theorem 5.2. $\square$

Acknowledgments. This work has been partially supported by grants PID2020-118281GB-C31 and MTM2017-86795-C3-1-P of Ministerio de Ciencia e Innovación, and 2017 SGR 1617 of Generalitat de Catalunya

REFERENCES

[1] R. Adler, A. Konheim and M. McAndrew, Topological entropy, *Trans. Amer. Math. Soc.* **114** (1965), 309–319.

[2] Ll. Alsedà, J. Guaschi, J. Los, F. Mañosas and P. Mumbrú, Canonical representatives for patterns of tree maps, *Topology* **36** (1997), no. 5, 1123–1153.

[3] Ll. Alsedà, D. Juher and F. Mañosas, Topological and algebraic reducibility for patterns on trees, *Ergodic Theory Dynam. Systems* **35** (2015), 34–63.

[4] Ll. Alsedà, D. Juher and F. Mañosas, On the minimum positive entropy for cycles on trees, *Transactions Amer. Math. Soc.* **369**(1) (2017), 187–221.

[5] Ll. Alsedà, J. Llibre and M. Misiurewicz, Combinatorial dynamics and entropy in dimension one, volume 5 of *Advanced Series in Nonlinear Dynamics*, World Scientific Publishing Co. Inc., River Edge, NJ (2000).

[6] Ll. Alsedà and X. Ye, No division and the set of periods for tree maps, *Ergod. Th. & Dynam. Sys.* **15** (1995), 221–237.

[7] S. Baldwin Generalizations of a theorem of Sarkovskii on orbits of continuous real-valued functions, *Discrete Math.* **67** (1987), no. 2, 111–127.

[8] F. Blanchard, E. Glasner, S. Kolyada and A. Maas, On Li-Yorke pairs, *J. Reine Angew. Math.* **547** (2002), 51–68.

[9] L. Block, J. Guckenheimer, M. Misiurewicz, L.S. Young, Periodic points and topological entropy of one-dimensional maps, *Global theory of dynamical systems*, SLNM **819** (1980), 18–34, Springer, Berlin.

[10] A. Blokh, Trees with snowflakes and zero entropy maps, *Topology* **33** (1994), 379–396.

[11] A. Blokh, R. Fokkink, J. Mayer, L. Oversteegen and E. Tymchatyn, Fixed point theorems in plane continua with applications, *Mem. Amer. Math. Soc.* **224** (2013), no. 1053, xiv+97.

[12] T.-Y. Li and J. Yorke, Period three implies chaos, *American Mathematical Monthly* **82** (1975), 985–992.

[13] J. Llibre and M. Misiurewicz, Horseshoes, entropy and periods for graph maps, *Topology* **32** (1993), 649–664.

[14] M. Misiurewicz, Minor cycles for interval maps, *Fundamenta Mathematicae* **145** (1994), 281–304.

[15] M. Misiurewicz and Z. Nitecki, Combinatorial patterns for maps of the interval, *Mem. Amer. Math. Soc.* **94** (1991), no. 456, vi+112.

[16] A. N. Sharkovsky, Co-existence of cycles of a continuous mapping of the line into itself, *Ukrain. Mat. Ž.* **16** (1964), 61–71. English translation: Proceedings of the Conference "Thirty Years after Sharkovsky's theorem: New perspectives" (Murcia, 1994). *Internat. J. Bifur. Chaos* **5** (1995), 1263–1273.

[17] P. Štefan, A theorem of Šarkovskii on the existence of periodic orbits of continuous endomorphisms of the real line, *Comm. Math. Phys.* **54** (1977), 237–248.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: alseda@mat.uab.cat
E-mail address: david.juher@udg.edu
E-mail address: manyosas@mat.uab.cat