Undominated rules with three alternatives in an almost unrestricted domain *

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Abstract

We consider the collective decision problem of a society choosing among three alternatives on a strict preference domain in which one preference ordering over alternatives is not admissible. We propose the family of *Sequential Pareto Undominated Rules* and characterize one of them as the unique full range, anonymous, tops-only, and strategy-proof voting rule.

Keywords: Strategy-proofness, anonymity, tops-onlyness, undominated set.

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1 Introduction

The French presidential election is a run-off voting system in which, if no candidate obtains an absolute majority in the first round, a second round of voting is held between the two top candidates. In 2002 presidential election, in the first round the conservative candidate Jacques Chirac received 19.88% of the votes, Jean-Marie Le Pen, leader of the far-right National Front Party, obtained 16.86%, Lionel Jospin, leader of the Social Party, obtained 16.18%, and a number of other candidates received a much smaller fraction of votes. The run-off was therefore between Jacques Chirac and Jean-Marie Le Pen and the former candidate won by a very large margin with 82.2% of the votes. Restricting the attention on the first three candidates it is fair to say from the election outcome that the preference ordering Jospin preferred to Le Pen preferred to Chirac was quasi-non-existent: Le Pen did not really progress and all the voters who had selected Jospin at the first stage voted Chirac in the run-off.[1]

Social choice theory establishes a clear difference between voting over two and three or more alternatives. Majority voting is a simple and effective rule to use in case of two alternatives, because no citizen or group of citizens have incentives to manipulate, expressing a preference different than their true one. In case of three or more alternatives no rule except the dictatorial one keeps this desirable property, unless some further assumptions are made on how citizens may rank the available alternatives.[2] In response to this fundamental result, a large literature has taken up the challenge of identifying domains on which possibility results emerge (Moulin, 1980, Barberà, Sonneschein, and Zhou, 1991, Nehring and Puppe, 2007). Still, in an arbitrary domain, it is not easy to check whether a rule is strategy-proof (no single individual has incentive to misreport her preferences) or group strategy-proof (no group of individuals have incentives to coordinate to jointly misreport their preferences). Consider for example, a collective decision problem inspired by our initial motivating example.

The citizens of a nation have to choose one among three alternatives: a, b and c. Citizens' preferences are strict and any ranking is admissible except the linear ordering $c \succ a \succ b$. All citizens who have c at the top of their preferences, prefer b to a. Notice that this is the largest domain of strict preferences with three alternatives in which there may exist a (group) strategy-proof voting rule. It follows that the set of

[1] We thank Maurice Salles for having suggested this illustrative example.

[2] See May (1952) and Gibbard (1973) and Satterthwaite (1975) as classical references for two and three or more alternatives, respectively.

individual preferences is given by five possible linear orderings:

$$\begin{matrix} c & b & b & a & a \\ b & c & a & b & c \\ a & a & c & c & b \end{matrix}$$

Suppose that alternative a is implemented unless there is an unanimous agreement that another alternative should be selected. Two different social choice functions may be proposed to reach a decision. According to the first voting rule (Rule A) the alternative c is chosen if and only if every citizen reports that c is the preferred alternative. If c is not the preferred alternative of every citizen, then the alternative b is chosen if and only if every citizen prefers b to a, otherwise a is chosen.

The second voting rule (Rule B) swaps alternative c with b: alternative b is chosen if and only if every citizen reports that b is the preferred alternative. If not, then the alternative c is chosen if and only if every citizen prefers c to a, otherwise a is chosen.

Do both voting rules provide the same incentives to report truthfully? The answer is negative, because Rule A is group strategy-proof while Rule B violates strategy-proofness (therefore, also violates group strategy-proofness).[3] To see why Rule B is not strategy-proof suppose that there are $n \geq 2$ citizens and consider the preference profile such that citizen 1's ranking over alternatives is $c \succ_1 b \succ_1 a$ and for every other citizen $i \neq 1$ the ranking is $b \succ_i a \succ_i c$. According to Rule B the alternative selected at this profile is a, but if citizen 1 manipulates and reports $b \succ_1 c \succ_1 a$, then b is chosen. It follows that the deviation is profitable for citizen 1. Rule B not only is manipulable, but it is also not efficient, because at the former preference profile in which a is chosen, a is Pareto dominated by b. Rule A is group strategy-proof (and therefore efficient). To see why, notice that if a is the preferred alternative of at least one citizen, then a is chosen and citizens with a different preferred alternative cannot change the outcome irrespectively of which preferences they report. If no citizen has a as preferred alternative, then all citizens prefer b to a and c is obtained only in the case where all citizens have c as their preferred alternative. Thus, in no cases an agent can change the chosen alternative.

In this paper we focus on the simplest case in which it is non-trivial to investigate whether a full range

[3]See Section 2 for a formal definition.

voting rule is strategy-proof, that is when there are three alternatives at stake. In addition to strategy-proofness we require that the voting rule satisfies two criteria that are standard in practice: anonymity, meaning that each agent's preference should play the same role in the decision process and tops-only property, implying that agents only report their preferred alternative and not the entire ranking. This second criterion is especially convenient jointly with strategy-proofness; there is empirical evidence (see for instance Hassidim et al., 2016) that individuals may to not understand or trust a strategy-proof mechanism; reporting a single alternative is much simpler than reporting a ranking and may help a cognitively limited agent to recognize that manipulation is useless. The rest of the paper is organized as follows. Section 2 presents the model, Section 3 illustrates the result and its proof.

2 Model

Let $A = \{a, b, c\}$ be the set of *alternatives* and $N = \{1, ..., n\}$ be a finite set of *agents*. Each agent i's preferences are linear orders over alternatives, denoted by P_i. For any x, $y \in A$, xP_iy denotes that x is strictly preferred to y by i at P_i. For any $i \in N$, let $\mathcal{D}_i = \mathcal{D}$ denote the set containing all strict preferences on A except the order cP_iaP_ib ("no-cab" domain). Let $t(P_i)$ denote the *top* alternative of P_i. Elements in $\mathcal{D}^n$ are called *preference profiles*, and are denoted by $P_N = (P_1, ..., P_n)$. We also write a preference profile P_N as $(P_C, P_{N\setminus C}) \in \mathcal{D}^n$ when we want to stress the role of coalition C in N. For any profile P_N, let $N_x(P_N) = \{i \in N : t(P_i) = x\}$ and $n_x(P_N)$ its cardinality, or N_x and n_x when no confusion about the profile arises. We say that an *alternative x is (Pareto) undominated at* $P_N \in \mathcal{D}^n$ if for any $y \in A\setminus\{x\}$ there exists $i \in N$ such that xP_iy. Given any $P_N \in \mathcal{D}^n$, let $U(P_N) \equiv \{x \in A : x$ is Pareto undominated at $P_N\}$ be *the set of all undominated alternatives* (equivalently, the set of Pareto efficient alternatives). The following two remarks point out some properties of the undominated set in our problem.

Remark 1. *Let $P_N \in \mathcal{D}^n$. (i) For each $x \in A$, if there exists an agent $i \in N$ whose top alternative is x, then $x \in U(P_N)$. (ii) if $a \in U(P_N)$ then $t(P_i) = a$ for some agent $i \in N$. (iii) $U(P_N) = \{x\}$ if and only if $t(P_i) = x$ for all agent $i \in N$.*

Two observations are in order. First, although two profiles of preferences have the same set of tops, the set of undominated alternatives can differ. Second, part *(ii)* of Remark 1 is false for b and c.[4]

Remark 2. *Let $P_N \in \mathcal{D}^n$ and ρ be a permutation of N, then $U(P_{\rho(N)}) = U(P_N)$.*

The proofs of these remarks are straightforward. The following remark highlights a key (unusual) feature of the "*no-cab*"domain concerning Pareto domination of alternative a.

We say that an *alternative x is (Pareto) dominated at $P_N \in \mathcal{D}^n$* if for some $y \in A\backslash\{x\}$, yP_ix for all $i \in N$.

Remark 3. *Let $P_N \in \mathcal{D}^n$. The following three statements are equivalent: (1) a is Pareto dominated at P_N. (2) a is Pareto dominated by b at P_N. (3) No voter $i \in N$ has a as top of P_i.*[5]

The equivalence of (1) and (3) is straightforward by parts (i) and (ii) of Remark 1. Clearly, (3) implies (2) and (2) implies (1).

A *voting rule* (or simply a *rule*) on $\mathcal{D}^n$ is a function $f : \mathcal{D}^n \to A$, that is, f selects an alternative for each preference profile. We impose several properties on our voting rules. The first one is *full range.*
A voting rule f on $\mathcal{D}^n$ is **full range** if for each $x \in A$, there exists $P_N \in \mathcal{D}^n$ such that $f(P_N) = x$.
The second property is *strategy-proofness*, that is, agents do not gain by misrepresenting their preferences.
A voting rule f on $\mathcal{D}^n$ is *manipulable at $P_N \in \mathcal{D}^n$ by coalition $C \subseteq N$* if there exists $P'_C \in \times_{i \in C}\mathcal{D}_i$ such that $P'_i \neq P_i$ and $f(P'_C, P_{N\backslash C})P_i f(P_N)$ for all $i \in C$.[6]
A voting rule f is **group strategy-proof** if it is not manipulable at any P_N by any coalition C. A voting rule f is **strategy-proof** if it is not manipulable at any P_N by any singleton coalition.
Notice that individual and group strategy-proofness are equivalent when there are only three alternatives (see Proposition 1 and Theorem 1 in Barberà, Berga, and Moreno, 2010). We impose two additional properties that are standard in the literature: anonymity and tops-only property. *Anonymity* ensures

[4]To show the first observation, let P_N and P'_N be such that they have the same set of top alternatives: a and c. In P_N some agents with top a prefer b to c but others c to b, while in P'_N all agents with top a prefer c to b. Thus, $U(P_N) = \{a, b, c\}$ and $U(P'_N) = \{a, c\}$. To show the second observation, note that $b \in U(P_N)$ but no agent consider b as the top alternative. To see that c can belong to the set of undominated alternatives of a profile in which no agent has c as top alternative, consider the profile P''_N with three agents such that $aP''_1bP''_1c$, $aP''_2cP''_2b$ and $bP''_3cP''_3a$.

[5]We thank an anonymous referee for suggesting this remark that simplifies the proof of the main result.

[6]Note that one could require all agents to gain but allow for some of them not to change their preferences. That would turn out to be equivalent to the definition of manipulation we use.

that the names of the agents do not matter.

A voting rule f on $\mathcal{D}^n$ is **anonymous** if for every permutation ρ of N and each preference profile $(P_1, P_2, ..., P_n) \in \mathcal{D}^n$ we have that $f(P_{\rho(1)}, P_{\rho(2)}, ..., P_{\rho(n)}) = f(P_N)$.[7]

Tops-only property guarantees that the voting rule is sensitive only to information about agents' top alternatives.

A voting rule f on $\mathcal{D}^n$ is **tops-only** if for any P_N, $P'_N \in \mathcal{D}^n$ such that $t(P_i) = t(P'_i)$ for all $i \in N$, then $f(P'_N) = f(P_N)$.

Weymark (2008) presents two well-known frameworks where tops-only is implied by strategy-proofness: single-peaked preferences and the standard continuous preferences in consumption. Note that this implication is not general. In particular, it does not hold in our environment as we show by means of Example 1 below.

3 Results and Discussion

In this section we present first a family of rules named Sequentially Pareto Undominated Rules, *SPUR*. Each rule is associated with a given linear order of alternatives and selects for each preference profile the first Pareto undominated alternative according to the assigned linear order. Formally:

Definition 1. *Fixed a linear order $\succ$ of alternatives. A voting rule f is the sequentially Pareto undominated rule for $\succ$, and we write $\succ$-**SPUR**, if for each $P_N \in \mathcal{D}^n$, $f(P_N) = x$ if and only if either $U(P_N) = \{x\}$, or else $x \in U(P_N)$ and for all $z \in U(P_N) \setminus \{x\}$, $x \succ z$.*

It is worth noticing that Rule A and Rule B in the motivating example described in the introduction are two SPURs: rule A is the SPUR with the associated linear order $a \succ b \succ c$ while rule B is the SPUR with the associated linear order $a \succ c \succ b$.

We now present two lemmas that will be useful for the proof of our main result.

Lemma 1. *Let f be an anonymous and tops-only voting rule. For any P_N, $P'_N \in \mathcal{D}^n$ such that for all $x \in A$, $n_x(P_N) = n_x(P'_N)$ then $f(P_N) = f(P'_N)$.*

[7]Note that for any profile and any permutation of agents the permuted profile, say $(P_{\rho(1)}, P_{\rho(2)}, ..., P_{\rho(n)}) \in \mathcal{D}^n$ is in the domain.

Proof. Note that by definition $P'_N = \left(P'_{N_h(P'_N)}, P'_{N_r(P'_N)}, P'_{N_s(P'_N)}\right)$. By anonymity, $f(P'_N) = f(P''_N)$ where $P''_N \equiv \left(P'_{\rho\left(N_h(P'_N)\right)}, P'_{\rho(N_r(P'_N))}, P'_{\rho(N_s(P'_N))}\right)$ where ρ is a permutation of N such that $\rho\left(N_h(P'_N)\right) = N_h(P_N)$, $\rho(N_r(P'_N)) = N_r(P_N)$, $\rho(N_s(P'_N)) = N_s(P_N)$. Note that ρ is well-defined since, by hypothesis, for each x, the cardinality of the sets $N_x(P'_N)$ and $N_x(P_N)$ is the same. Moreover, by tops-onlyness, $f(P_N) = f(P''_N)$ since for each alternative x and each agent $i \in N_x(P_N)$, $t(P_i) = t(P''_i) = x$. □

Thanks to Lemma 1, from now on, we may only refer to the cardinality of the set of agents having the same top alternative instead of indicating the full set of such agents.

The second lemma tells us that if c is the outcome of the voting rule at some profile, it remains the outcome at any profile in which the number of agents with c as top remains the same.

Lemma 2. *Let f be a full range, anonymous, tops-only, and strategy-proof voting rule. For any $P_N \in \mathcal{D}^n$ such that $f(P_N) = c$ and any $P'_N \in \mathcal{D}^n$ such that $n_c(P'_N) = n_c(P_N)$ then $f(P'_N) = f(P_N)$.*

Proof. Let f be an anonymous, tops-only, and strategy-proof voting rule. Let $P_N \in \mathcal{D}^n$ be such that $f(P_N) = c$ and $P'_N \in \mathcal{D}^n$ such that $n_c(P'_N) = n_c(P_N)$. Let $\widetilde{P}_N$ be any profile such that $n_x(\widetilde{P}_N) = n_x(P_N)$ for all $x \in A$ and agents in $N_a(\widetilde{P}_N) \cup N_b(\widetilde{P}_N)$ have c as the worst alternative. Note that $\widetilde{P}_N$ is well-defined. Moreover, by anonymity and tops-onlyness (by Lemma 1), $f(\widetilde{P}_N) = f(P_N) = c$. Consider the set of profiles such that for each P''_N in this set $n_c(P_N) = n_c(P''_N) = n_c(\widetilde{P}_N)$. By group strategy-proofness and anonymity, $f(P''_N) = f(\widetilde{P}_N) = c$. There exists P''_N in this set such that for all $i \in N$, $t(P'_i) = t(P''_i)$. By tops-onlyness $f(P'_N) = f(P''_N) = c$. □

We are now able to state our main result characterizing the unique full range voting rule satisfying anonymity, tops-onlyness, and strategy-proofness.

Theorem 1. *A full range voting rule f on $\mathcal{D}^n$ ("no-cab" domain) is anonymous, tops-only, and strategy-proof if and only if f is the $\succ$-SPUR with associated linear order $a \succ b \succ c$.*

Proof. Step 1. For any $P_N \in \mathcal{D}^n$, $f(P_N) \in U(P_N)$. This is straightforward by group strategy-proofness and full range:

Let $P_N \in \mathcal{D}^n$ and suppose to get a contradiction that $f(P_N) \notin U(P_N)$, that is, there exists $y \in A \backslash f(P_N)$

such that for all $i \in N$, $yP_i f(P_N)$. By full range, let P'_N be such that $f(P'_N) = y$. Note that N would manipulate f at P_N via P'_N, which is a contradiction to group strategy-proofness.

Step 2. For any $P_N \in \mathcal{D}^n$ such that $\#U(P_N) \geq 2$, then $f(P_N) \neq c$.

Let $P_N \in \mathcal{D}^n$ such that $\#U(P_N) \geq 2$. Suppose by contradiction that $f(P_N) = c$. By Step 1, $c \in U(P_N)$.

Case 1: If $n_c(P_N) = 0$, by tops-onlyness, $f(P'_N) = f(P_N)$ for P'_N such that for each $i \in N$, $t(P'_i) = t(P_i)$ and c is the worst alternative. Notice that c is dominated in P'_N, thus we have a contradiction to Step 1.

We now distinguish two cases where $n_c(P_N) > 0$.

From now on, when we use n_x we refer to $n_x(P_N)$.

Case 2: $n - n_c \geq n_c$. Let P'_N be such that there are n_c agents with c as top and $n - n_c$ with b as top. By Lemma 2, $f(P'_N) = c$. Change, one by one, the preferences of each agent in $N \backslash N_c(P'_N)$ to a preference where his top is c until the number of agents having c as top is $n - n_c$. By strategy-proofness, in each one of the steps the outcome selected is c. Let P''_N be the profile obtained after all these changes where note that there are n_c agents with top b and $n - n_c$ with top c. By strategy-proofness, $f(P''_N) = c$.

By tops-onlyness, without loss of generality, in P''_N, the n_c agents with b as top can have c as the worst alternative. Let $\widetilde{P}_N$ be such that there are n_c agents with a as top and $n - n_c$ with c as top. By Lemma 2, $f(\widetilde{P}_N) = f(P''_N) = c$.

Let P'''_N be such that there are n_c agents with a as top and $n - n_c$ with b as top. Suppose that $f(P'''_N) = b$. By tops-onlyness, without loss of generality, in P'''_N, the n_c agents with a as top can have b as the worst alternative. These agents by announcing c as top alternative can obtain $c = f(P'_N)$ which is a contradiction to group strategy-proofness.

Suppose now that $f(P'''_N) = a$. By tops-onlyness, without loss of generality, in P'''_N, the $n - n_c$ agents with b as top can have a as the worst alternative. These agents by announcing c as top alternative can obtain $c = f(\widetilde{P}_N)$ which is a contradiction to group strategy-proofness.

Suppose now that $f(P'''_N) = c$. By tops-onlyness, without loss of generality, in P'''_N, all agents have c as the worst alternative. Since c is dominated, we get a contradiction to Step 1. We get the desired contradiction since $f(P'''_N)$ is not defined, thus f is not well-defined. This ends the proof of Case 2.

Case 3: $n - n_c < n_c$. Take any $i \in N_c(P_N)$. Consider P'_i such that $t(P'_i) = x$ where $n_x = \min\{n_b, n_a\}$. Without loss of generality, suppose that $x = b$. Let $P'_N = (P'_i, P_{N \backslash \{i\}})$. We show that $f(P'_N) = c$.

Otherwise, if $f(P'_N) = a$ then, by tops-only property $f(P''_i, P_{N\backslash\{i\}}) = a$ for P''_i such that $t(P''_i) = b$ and a is the worst alternative. Note that this contradicts strategy-proofness since agent i would manipulate f at $(P''_i, P_{N\backslash\{i\}})$ via P_i. If $f(P'_N) = b$ then, by tops-only property $f(P'_i, P'_j, P_{N\backslash\{i,j\}}) = b$ such that $t(P'_j) = a = t(P_j)$ and b is the worst alternative for $j \in N\backslash\{i\}$. By Lemma 2, alternative c is chosen in the profile $(P'_i, P''_j, P_{N\backslash\{i,j\}})$ where $t(P''_j) = c$, since the number of agents having c as top in $(P'_i, P''_j, P_{N\backslash\{i,j\}})$ is the same as in P_N. Note that this contradicts strategy-proofness since agent j would manipulate f at $(P'_i, P'_j, P_{N\backslash\{i,j\}})$ via P''_j. Then, $f(P'_N) = f(P'_i, P_{N\backslash\{i\}}) = c$.

For P'_N note first that $n_c(P'_N) = n_c - 1$ and $n - n_c(P'_N) = n - n_c + 1$. Redefine P_N as P'_N and go to Step 2.

If $n_c - 1 = 0$ we get a contradiction by Case 1. If $0 < n_c - 1 \leq n - n_c$, then we get a contradiction by Case 2. Otherwise, if $n_c - 1 > n - n_c$, we can repeat the same argument in Case 3 for each profile that we obtain changing one by one agents with top c until we reach a contradiction using Cases 1 or 2.

<u>Step 3</u>. For any $P_N \in \mathcal{D}^n$ such that $a \in U(P_N)$, then $f(P_N) = a$.

By contradiction, let $P_N \in \mathcal{D}^n$ be such that $a \in U(P_N)$, but $f(P_N) \neq a$. By Steps 1 and 2, $b \in U(P_N)$ and $f(P_N) = b$. Since $a \in U(P_N)$, we get that $n_a > 0$ by part (ii) of Remark 1. Let P'_N be any profile such that for each agent $i \in N_c(P_N)$, $t(P'_i) = b$ while for any agent $j \in N\backslash N_c(P_N)$, $P'_j = P_j$. By group strategy-proofness, $f(P'_N) = b$ (otherwise, if $f(P'_N) \in \{c, a\}$ agents in $N_c(P_N)$ would manipulate f at P'_N via $P_{N_c(P_N)}$). Consider a profile P''_N that differs from P'_N only for agents in $N_a(P'_N)$ that change their preferences such that a is still the top while b is the worst alternative. By tops-only property, $f(P''_N) = b$. Now, consider a profile P'''_N that differs from P''_N only for agents $i \in N_b(P''_N)$ who change their preferences such that c is now their top. By group strategy-proofness, $f(P'''_N) \neq a$ (otherwise, $N_b(P''_N) = N_c(P'''_N)$ would manipulate f at P'''_N via $P''_{N_b(P''_N)}$). By Step 1 above, $f(P'''_N) \neq b$ since $b \notin U(P'''_N) = \{a, c\}$. Moreover, by Step 2, $f(P'''_N) \neq c$ since $\#U(P'''_N) = 2$. Thus, we have a contradiction.

To finish the only if part, notice that using Steps 1 to 3 and Remark 3 the voting rule f must be the $\succ$-SPUR for $a \succ b \succ c$, which can be described by the following algorithm: consider any P_N

(1) if a is the top alternative for some $i \in N$ then $f(P_N) = a$, otherwise go to (2);

(2) if b is the top alternative for some $i \in N$ then $f(P_N) = b$, otherwise $f(P_N) = c$.

Finally, it remains to prove the "if" part, that is, the $\succ$-SPUR for $a \succ b \succ c$ is group strategy-proof, tops-only, and anonymous. By construction, the full range rule is tops-only and anonymous.

To check (group) strategy-proofness of f notice that if a is the top alternative of at least one agent i at profile P_N, then the outcome is alternative a. No agent $j \neq i$ can change the outcome. Suppose that no agent has a as top alternative. If b is the top alternative of at least one agent, say i, then the outcome is b. By Remark 3, a is Pareto dominated by b at P_N. Thus, no agent has incentives to deviate to get a. Moreover, no agent $j \neq i$ can change the outcome and get c. Suppose that no agent has either a or b as top alternative. Then, $f(P_N) = c$ and since all agents have c as top alternative, none has incentives to deviate. □

The result in our Theorem 1 is robust as we show below presenting examples of full range rules satisfying every pair of properties but not the third one.

Example 1. *The full range voting rule $\succ$-SPUR with the associated linear order $b \succ c \succ a$, say f, is strategy-proof and anonymous but it is not tops-only.*

Anonymity of f is straightforward by Remark 2.

We now show that f is strategy-proof: if b is the preferred alternative for at least one agent, then $b \in U(P_N)$ (by part (i) of Remark 1), thus b is chosen by definition of f, and agents with a different preferred alternative cannot change the outcome irrespective of the preference they report.

If no agent has b as preferred alternative, either (i) all agents have the same preferred alternative, or (ii) for some agents c is the preferred alternative (and all them have preferences $cPbPa$) and for some other agents a is the preferred alternative. In the former case (i): by part (iii) of Remark 1 this alternative is the only undominated alternative, and thus it is chosen (obviously no agent has incentives to manipulate since everyone gets his best alternative). In the latter case (ii), we distinguish two subcases:

(1) if some agent with top a has also c as the bottom alternative, and

(2) if all agents with top a have b as bottom alternative.

For case (1), $b \in U(P_N)$ and thus b is chosen.

Note that for this case the only coalitions that might want to manipulate are those formed by agents either (a) with preferences $aPbPc$ and $aPcPb$ if they could get a, or (b) with preferences $aPcPb$ and $cPbPa$, if they could get c.

Note that situation (a) can not happen because the outcome cannot be a since there are agents with c as top. Thus, either b or c could be the outcome.

Situation (b) can not happen because to get c agents should change preferences such that $b \notin U(P'_N)$ and $c \in U(P'_N)$. Thus, for the former to hold, preferences in P'_N of agents in the deviating coalition can be neither $cPbPa$, $bPaPc$ nor $bPcPa$. Thus, their preferences in P'_N must have a as top, implying that $c \notin U(P'_N)$ contradicting the latter.

For case (2), $b \notin U(P_N)$, $c \in U(P_N)$ and thus, c is chosen.

Note that agents with preferences $cPbPa$ get their preferred alternative and they do not want to deviate. Only a coalition of agents with preferences $aPcPb$ might want to manipulate if they could get a. The latter is impossible since in the new profile c would be undominated.

To show that f violates tops-only property, let P_N and P'_N be such that they have the same set of top alternatives: a and c. In P_N some agents with top a prefer b to c but others c to b, while in P'_N all agents with top a prefer c to b. Since $U(P_N) = \{a, b, c\}$ and $U(P'_N) = \{a, c\}$, by definition of f, we get that $f(P_N) = b$ but $f(P'_N) = c$.

Example 2. *The full range voting rule described by the following algorithm is tops-only and anonymous but it is not strategy-proof: consider any P_N*

(1) if a is the top alternative for some $i \in N$ then $f(P_N) = a$, otherwise go to (2);

(2) if c is the top alternative for some $i \in N$ then $f(P_N) = c$, otherwise $f(P_N) = b$.[8]

By construction this rule is anonymous and tops-only. Also observe that this voting rule is not strategy-proof because it coincides with Rule B defined in the introduction.

Example 3. *Let f be a voting rule where agent $i \in N$ is a dictator (that is, $f(P_N) = t(P_i)$ for all $P_N \in D^n$). Any dictatorial rule has full range, is tops-only and strategy-proof but it is not anonymous.*[9]

In the problem of the provision of a public good where agents have single-peaked preferences, the so-called generalized Condorcet-winner solutions are characterized by Moulin (1980), Barberà and Jackson (1994), and Ching (1997) on the basis of strategy-proofness. All of them are tops-only (see also Weymark,

[8] This rule is the $\succ$-SPUR with associated linear order $a \succ c \succ b$.

[9] The proof is straightforward and left to the reader.

2011).[10] We would like to observe that the rule we characterize in Theorem 1 can be also rewritten and interpreted as a member of the generalized Condorcet-winner solution. To do that, we fix the following order of alternatives $a \succ b \succ c$. Then, the rule chooses the highest alternative among agents's top alternatives relative to this order. It is worth mentioning that in the case with three alternatives if we depart from the set of single-peaked preferences and we add one preference to obtain the "no-cab" domain, the unique anonymous rule in the class of generalized Condorcet-winner solutions is the $\succ$-SPUR with associated linear order $a \succ b \succ c$ that we characterize in Theorem 1.

[10]Thomson (1993, 1999) characterized a subclass of the above family of rules using Pareto efficiency and welfare-domination under preference-replacement, which are also anonymous.

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