

The Optimal Selective Logging Regime and the Faustmann Formula

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Abstract

This study analyzes the optimal selective logging regime of a size-distributed forest where individual trees compete for scarce resources such as space, light, and nutrients. The decision problem of the forest manager is formulated as a distributed optimal control problem. The interpretation of the first-order conditions allows a generalization of the Faustmann formula. In an empirical part, this article numerically determines the optimal management regime of a size-structured forest and shows that the optimal selective logging regime is associated with a normal forest under a wider variety of situations than stated in the previous literature.

Key words: distributed optimal control; density effects; forest management; Faustmann rule; selective logging

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1 Introduction

Frequently, forest management is based on the Faustmann or maximum sustained yield rules, which determine the optimal cutting age of the trees. Since these criteria are based on the assumption that all trees have the same age, their application leads to the clear cutting of the entire stand. This management regime has been criticized because it pays no attention to the ecological value, in the form of biological diversity in forest land (Doyon et al., 2005; Sawadogo et al., 2005), and scenic or recreational values (Scarpa et al., 2000). Since trees in a forest provide food, shelter and substrate for other life forms, maintaining a rich variety of trees conserves biodiversity in a forest ecosystem (Lin and Buongiorno, 1998) and improves forest attributes associated with scenic and recreational values of the forest land. For example, a complex tree and canopy structure helps to provide preferable habitat for wildlife (Hunter, 1990; Whittam et al. 2002).

To overcome the negative effects of the clear cutting regime, tree selection silviculture (uneven-aged management), for which there is a very strong interest in temperate as well as in tropical regions, has been proposed (Vanclay, 1995). Managers of public forests are likely to have a mandate to consider not only the monetary benefits to be obtained from timber production but also the benefits to be gained from biodiversity conservation, and from maintenance of forest land with high scenic and recreational values. In contrast, private landowners and timber companies might, *a priori*, be less interested in adopting a selective logging regime since they are more inclined to focus exclusively on the monetary benefits of timber production. However, the fact that an increasing number of private owners and timber companies certify their private forest holdings by using standards developed, for instance, by the Forest Stewardship Council, the Canada Standard Association or the Pan European Forest Certification Council makes it more likely that private owners and timber companies will also adopt a selective logging regime. Despite these concerns, the economic literature has described the characteristics of the optimal selective logging regime only in part.

The previous literature emphasizes the introduction of an additional independent variable of the choice functions. Besides calendar time, a structuring variable of the trees, either age or diameter, was considered.

25 Several empirical studies have introduced the structuring variable diameter of the tree to resolve the problem
26 of optimal management of the uneven-aged forest. Initially, Adams and Ek (1974) determined only the
27 optimal long-run allocation of all trees among the different tree vintages, but not the optimal replacement
28 periods. Haight et al. (1985) and Lin and Buongiorno (1998) solved simultaneously for the optimal long-run
29 allocation of all trees among the different tree vintages, and the optimal replacement periods. These studies
30 were formulated in a discrete framework, i. e., time and diameter can only take on certain values, and,
31 consequently, they cannot provide an analytical formulation of the optimal selective logging rule. Although
32 later studies by Adams et al. (1996), and Sedjo and Lyon (1990) utilized discrete time and a continuous
33 structuring variable for their policy analysis, they did not establish an analytical formulation for the optimal
34 logging pattern.

35 Moreover, several quite remarkable theoretical studies did provide a rule for the optimal selective logging
36 regime by analyzing the properties of the transient path and of the steady state (Mitra and Wan, 1985; Mitra
37 and Wan, 1986; Wan, 1994; Salo and Tahvonon, 2002; Uusivuori and Kuuluvainen, 2005; Cunha e Sa
38 et al., 2007). However, these articles are based on a stylized analytical model where the growth process
39 of the biomass is not modeled explicitly. Every year trees pass from the current to the sequent age class.
40 As each age class is associated with a particular biomass that increases with age, trees “grow” as they get
41 older. Hence, the current theoretical models do not establish a functional relationship between biomass and
42 age, but rather form a tuple of different pairs of biomass and age. In this respect, their analytical results
43 are limited to certain points of combinations of age and biomass, but do not fully describe the underlying
44 biophysical processes proposed in the literature of biological mathematics (Keyfitz, 1968; Keyfitz, 1977).

45 The previous theoretical literature did not explicitly model competition between individuals belonging to the
46 same population. Individuals compete for space or light and nutrients. In the absence of predators, the life
47 cycle of a single individual is mainly influenced by the other individuals of the population. Thus, in order
48 to model biological growth, mortality and reproduction processes correctly, one does not only have to take

49 into account individual characteristics such as age but also the distribution of the individual characteristics
50 over the entire population. Given the same initial amount of biomass, the growth rate of the forest will be
51 higher as more young individuals form part of it. In other words, the more young trees there are, the less
52 competition individual trees face and therefore the growth rate of the entire forest increases with the share
53 of young trees.

54 The reason why growth processes based on continuous time and a continuous structuring variable have
55 not been considered may reside in the fact that they lead to distributed optimal control problems where
56 the law of motion is given by partial differential equations (PDE), or partial integrodifferential equations¹
57 if density effects are considered. The previous theoretical and empirical literature usually describes the
58 evolution of an age (size) distributed forest by a set of difference equations in time for a number of age (size)
59 cohorts. These difference equations are an approximation (discretization over time and age (cohorts)) of the
60 continuous evolution of the forest over time and age. The previously employed discretization procedure
61 seems reasonable and has been practiced widely; however, it is not known whether the set of difference
62 equations describes the evolution correctly. The set of difference equations is set up ad hoc and is not derived
63 from the underlying partial differential equation. Notice, for instance, that the set of difference equations in
64 the paper by Haight (1987) is identical for an age and size structured population. However, the PDEs for
65 age or size-structured populations are not identical. The PDE of a size-structure population incorporates a
66 function that relates time and size and therefore is not present in the PDE for an age-structured population.²
67 On the other hand, partial differential or integrodifferential equations allow not only the age but also the size
68 of the trees to be considered. In fact, the size of the trees determines their economic value. As discussed
69 below, age can only be considered as a poor proxy for the size of a tree, and therefore, any analysis based
70 on age instead of size may be misleading. Moreover, PDEs permit the price of timber to be formulated as
71 a function of tree size, and to incorporate density effects. Consequently, considering competition between

¹The unknown function of the PDE is a function of time and age (size). In cases where an integral over the unknown function forms part of a partial differential equation, the PDE becomes a partial integrodifferential equation.

²To be more precise this function is equal to one for an age-structured population since time and age move with the same speed.

72 individuals, continuous time, and a continuous structuring variable allows the definition of an analytical
73 formulation of the optimal harvesting rule for selective logging, and its comparison with the Faustmann
74 rule, which is based on continuous time.

75 Unfortunately, if the forest dynamics are governed by a partial integrodifferential equation it is not possible
76 to obtain a complete analytical solution of the optimal forest management regime and one has to resort to
77 numerical techniques. In contrast to the existing literature where distributed control problems are solved
78 numerically by a gradient projection method (Feichtinger et al., 2004) or Markov decision process models
79 (Lin and Buongiorno, 1998), we employ a different technique known as the “Escalator Boxcar Train”. This
80 technique has the advantage of not requiring programming numerical algorithms since it can be implemented
81 with standard optimization packages such as GAMS. Moreover, it simplifies the estimation of biological
82 equations used in the model as it will be shown in the empirical study. A presentation of the numerical
83 method used in this article can be found in Goetz et al. (2008).

84 This article presents a theoretical model that describes the characteristics of the optimal selective-logging
85 regime of a size-distributed forest that maximizes timber net benefits. The law of motion of the economic
86 model is governed by a partial integrodifferential equation that describes the evolution of the forest stock
87 over time. This formulation allows size dependent economic variables and modeling biophysical processes
88 to be taken into account more realistically. As an extension of the literature this article provides an ana-
89 lytical formulation of the optimal selective-logging rule whose interpretation allows generalization of the
90 Faustmann formula. In an empirical part the article determines the optimal selective-logging regime of a
91 size-distributed forest from a private perspective, and conducts a sensitivity analysis. It shows that the cycli-
92 cal evolution of the variables vanishes along the optimal path giving rise to a non-cyclical logging pattern
93 and a fairly uniform diameter distribution of the trees. This distribution provides a constant flow of income
94 and is known as a normal forest. In comparison with the previous theoretical literature where a normal forest
95 only emerged under special conditions, we find that the standard result is the emergence of a normal forest.

96 The article is organized as follows. Section 2 describes the features of the model. As such it is divided into
97 one subsection that describes the underlying biological processes and another that outlines the economic
98 decision problem in the form of a distributed optimal control problem. Section 3 shows that the optimal
99 selective cutting rule for a size-distributed forest can be interpreted as a generalization of the Faustmann for-
100 mula for even-sized forests, and analyzes the implications of a steady-state on the optimal forest distribution.
101 The next section determines the optimal selective cutting regime in an empirical setting, and analyzes the
102 changes in the long run outcome as a result of a change in different parameters. Finally, section 5 presents
103 the conclusions.

104 **2 The model**

105 Before presenting the complete economic model that will allow us to determine the optimal selective logging
106 regime, we have characterized the underlying biological model that describes the growth process of the trees.

107 **2.1 The growth process of the trees**

108 In previous theoretical economic literature the age of the tree is regarded as the structuring variable of the
109 biological population (Salo and Tahvonen, 2002; Wan, 1994). In biological science, however, it is usu-
110 ally not the age but physiological or behavioral characteristics, for instance, size, life cycle stages, gender
111 or genetic differences, or behavioral activities, that determine the biological life cycle of the individual
112 (Cushing, 1988). Likewise, from an economic point of view it is not the age but the size of the tree that
113 is important. The price of lumber changes with the size of the tree but not with age. And, as established
114 by forest scientists, the age of a tree is not very closely related to its size (Björklund, 1999). Large genetic
115 variety between the trees, and between the different local conditions of each tree makes it difficult to estab-
116 lish a functional relationship between size and age. Consequently, the age-size relationship resulting from
117 an econometric estimation is very poor (Seymour and Kenefic, 1998). In forestry, the size of a tree, and

118 consequently the size of a forest, is usually measured by the diameter at breast height, i. e., the diameter of
 119 the trunk at a height of 1.30 m above the ground. We denote the diameter of a tree by $l \in \Omega$, $\Omega \equiv [l_0, l_m)$,
 120 where l_0 and l_m indicate the biological minimum and maximum size of a tree. The exogenous variable l
 121 together with calendar time t form the domain of the control and state variables. We assume that a diameter-
 122 distributed forest can be fully characterized by the number of trees and by the distribution of the diameter
 123 of the trees. In other words, space and the local environmental conditions of the trees are not taken into
 124 account. Given that the diameter value of a tree lies in the interval $[l_0, l_m)$, and that the number of trees is
 125 large by assumption, the distribution of the trees can be represented by a density function, denoted by $x(t, l)$,
 126 which indicates the population density with respect to the structuring variable, l , at time t . Therefore, the
 127 number of trees in the forest at time t is given by

$$128 \quad X(t) = \int_{l_0}^{l_m} x(t, l) dl. \quad (1)$$

129 The forest dynamic is driven by the processes of reproduction, growth, and mortality, which in turn are
 130 influenced by environmental conditions. Let us define $g(E(t), l)$ the rate of change in the diameter of a tree,
 131 as a function of its current diameter l , where $E(t)$ presents a collection of environmental characteristics that
 132 affect the growth rate of the individual tree. In the absence of predators, these environmental characteristics
 133 are given by the local conditions where the tree is growing, and by the neighboring trees. The local con-
 134 ditions and the competition between individual trees for space, light, and nutrients affect the life cycle of
 135 each tree. Since our model does not consider space, the variable $E(t)$ presents exclusively the competition
 136 between individuals.³ Environmental characteristics $E(t)$ can be expressed by the total number of trees, or
 137 the basal area of the stand. A large basal area of the stand signifies a high competition pressure on the life
 138 cycle of an individual tree that decreases the diameter growth (Trasobares et al., 2004). Therefore, we use

³Although the literature on forest economics contains complex optimization models where space is considered explicitly, their solutions are based on heuristic approaches and not on efficient optimization techniques (Weintraub et al., 2007).

139 the stand basal area to take account of the competition between individuals,⁴ that is,

$$140 \quad E(t) = \int_{l_0}^{l_m} \frac{\pi}{4} l^2 x(t, l) dl. \quad (2)$$

141 Hence, the change in the diameter over time of a single tree is given by

$$142 \quad \frac{dl}{dt} = g(E(t), l), \quad (3)$$

143 The instantaneous death rate is denoted by $\delta(E(t), l)$. It describes the rate at which the probability of
144 survival of a tree, given the environmental characteristics $E(t)$, decreases with time.

145 The reproduction of the forest can be modeled as internal reproduction or planting. In the former case we
146 would obtain a boundary condition for the partial integrodifferential equation that reflects the reproduction
147 process. Since seed production by individual trees is very high (Karlsson and Örlander, 2000), it is space,
148 light, and nutrients that are the limiting factors for the upgrowth of young trees, and not the reproduction
149 process itself. For this reason we can assume that the number of seeds that turn into seedlings is sufficiently
150 large. This allows the forest manager to choose the number of trees with a diameter of l_0 by removing any
151 additional trees. The number of upgrowing trees chosen is denoted by $p(t, l_0)$. In the case of planting, we
152 are dealing with a forest that is completely managed, where young trees with a diameter of l_0 are planted and
153 no biological reproduction takes place. Hence, for both reproduction systems the control variables for the
154 management of the forest are given by $u(t, l)$ and $p(t, l_0)$, denoting cutting density at time t with diameter
155 l , and the flux of the trees with diameter l_0 respectively. Young trees are either grown up to diameter l_0 , or
156 planted with diameter l_0 , at time t . Thus, the optimal forest management problem is a distributed optimal
157 control problem where the time dependent control variable $u(t, l)$ is distributed over l (Feichtinger and
158 Hartl, 1986). In contrast, the time dependent boundary control variable $p(t, l_0)$ refers only to the initial

⁴We assume that the trees with different sizes are distributed evenly over the stand. If this is not the case, other intra-specific competition indices that are distance dependent, such as the Lorimer area index (García-Abril et al., 2007), are more appropriate.

159 diameter l_0 of the tree. Although our modelling approach allows consideration of both planting and natural
 160 reproduction, we frame our analysis in the context of natural reproduction since employed selective logging
 161 regimes frequently rest on natural reproduction. Based on the well known McKendrick equation for age
 162 structured populations (McKendrick, 1926), the dynamics of the diameter-distributed forest can be described
 163 by the following partial integrodifferential equation discussed by de Roos (1997), or by Metz and Diekmann
 164 (1986)

$$165 \quad \frac{\partial x(t, l)}{\partial t} + \frac{\partial g(E(t), l) x(t, l)}{\partial l} = -\delta(E(t), l) x(t, l) - u(t, l) \quad (4)$$

166 subject to the boundary condition $g(E(t), l_0) x(t, l_0) = p(t, l_0)$. The two terms on the left-hand side of
 167 equation 4 present the change in the tree density over time and diameter; the second term not only considers
 168 the diameter but also takes the interdependence between diameter and time $\frac{\partial(gx)}{\partial l} = g \frac{\partial x}{\partial l} + \frac{\partial g}{\partial l} x$ into account,
 169 i.e., it presents the temporal change in diameter multiplied by the change in tree density over diameter plus
 170 the temporal change in diameter over diameter multiplied by the tree density.⁵ Hence, the flux of the tree
 171 density with respect to diameter and time has to equal the terms of the right-hand side of equation 4, given
 172 by the product of the mortality rate and the tree density, and the density of the logged trees.

173 2.2 The distributed optimal control problem

174 We assume that the forest is privately owned and managed over a planning horizon of t_1 . Using the defini-
 175 tions given in the preceding section, the formal decision problem of the forest owner can be stated as:

$$176 \quad \max_{u(t, l), p(t, l_0)} \int_0^{t_1} \int_{l_0}^{l_m} B(x(t, l), u(t, l)) e^{-rt} dl dt - \int_0^{t_1} C(p(t, l_0)) e^{-rt} dt \quad (D)$$

$$+ \int_{l_0}^{l_m} S^{t_1}(x(t_1, l)) e^{-rt_1} dl + \int_0^{t_1} S^{l_m}(x(t, l_m)) e^{-rt} dt,$$

177 subject to the constraints

⁵If the structuring variable was age, the function g would be constant and equal to 1 since the aging of the individual by one year corresponds to calendar time. In this case the term $\frac{\partial(gx)}{\partial l}$ would simply yield $\frac{\partial x}{\partial t}$.

$$\frac{\partial x(t, l)}{\partial t} \equiv -\frac{\partial(g(E(t), l) x(t, l))}{\partial l} - \delta(E(t), l) x(t, l) - u(t, l),$$

$$x(t_0, l) = x_0(l), \quad g(E(t), l_0) x(t, l_0) = p(t, l_0), \quad p(t, l_0) \geq 0, \quad u(t, l) \geq 0,$$

where $E(t)$ is given by equation (2), and r denotes the discount rate. The twice-differentiable function $B(x, u)e^{-rt}$ presents the discounted net benefits of the timber. It depends not only on the number of logged trees but also on the number of standing trees since it incorporates the maintenance cost of the forest. It is assumed that the maintenance cost function is concave, and thus $B_x < 0, B_{xx} < 0$. It is also assumed that the net benefit function is strictly concave in u . The twice-differentiable and strictly convex function $C(p)e^{-rt}$ expresses the discounted cost of nursing trees up to diameter l_0 , the differentiable function $S^{t_1}(x)e^{-rt_1}$ the discounted value of the standing trees at the final point in time of the planning horizon, and the differentiable function $S^{l_m}(x)e^{-rt}$ expresses the discounted value of the standing trees that have reached the maximum diameter l_m .⁶ The term $x_0(l)$ denotes the initial diameter distribution of the trees, and the restriction $g(E(t), l_0) x(t, l_0) = p(t, l_0)$ requires that the flux of the change in diameter at diameter l_0 multiplied by the tree density coincides with the total flux of the diameter of trees with diameter l_0 . Finally, the control variables must be nonnegative.

Let the costate variable related with the dynamics of the forest be denoted by $\lambda(t, l)$, and the Lagrange multiplier related with the restriction $g(E(t), l_0) x(t, l_0) = p(t, l_0)$ by $\lambda_0(t)$. Utilizing Theorem 2 of Hritonenko et al. (2008) the following necessary conditions can be obtained:

⁶In the case of planting, the function $C(p)$ expresses the cost of planting trees with diameter l_0 and obviously has to be specified differently to how it is expressed in the case of natural reproduction.

$$e^{-rt}B_u - \lambda(t, l) + \mu_1(t, l) = 0, \quad \forall t, \quad \forall l \quad (5)$$

$$-e^{-rt}C'(p(t, l_0)) + \lambda_0(t) + \eta(t) = 0, \quad \forall t \quad (6)$$

$$p(t, l_0) - g(E(t), l_0) x(t, l_0) = 0, \quad \forall t \quad (7)$$

194

$$\frac{\partial \lambda(t, l)}{\partial t} + g(E(t), l) \frac{\partial \lambda(t, l)}{\partial l} = + (r + \delta(E(t), l)) \lambda(t, l) - B_x - \frac{\pi}{4} l^2 \int_{l_0}^{l_m} \lambda(t, l) \left(\frac{\partial (g_E x(t, l))}{\partial l} + \delta_E x(t, l) \right) dl, \quad (8)$$

$$\frac{\partial x(t, l)}{\partial t} = - \frac{\partial (g(E(t), l) x(t, l))}{\partial l} - \delta(E(t), l) x(t, l) - u(t, l), \quad x(0, l) = x_0(l), \quad (9)$$

195 where μ_1 and η are Kuhn-Tucker multipliers related to the non-negativity constraints of the decision vari-
 196 ables u and p , respectively. For an interior solution the first necessary condition, equation (5) states that
 197 along the optimal path the discounted marginal net benefits of timber should equal the shadow price λ (*in*
 198 *situ* value) of the forest stock for every t and l . In contrast to lumped optimal control, distributed opti-
 199 mal control requires that this equation holds along the optimal path not only with respect to time, but also
 200 with respect to diameter. Thus, the forest manager maximizes his/her benefits not only over time but also
 201 with respect to diameter at every instant of time. In other words, the forest manager practices selective
 202 logging. Equation (6) states that the discounted marginal cost of nursing trees up to diameter l_0 should
 203 equal, at every moment of time the future marginal benefits of this “tree”, i.e. the marginal net benefits
 204 that accrue from time t to t_1 . Hence, corresponding with the first necessary condition, the forest man-
 205 ager to some extent also practices selective nursing by choosing the time and the number of trees to be
 206 grown, but not their diameter. Equation (7) reproduces the constraint associated with $\lambda_0(t)$ and reflects
 207 the fact that the increase in diameter of the trees has to coincide with the flux of ingrowing trees with
 208 diameter l_0 . Necessary condition (8) shows that the marginal change in the *in situ* value over time and
 209 diameter has to equal the sum of the forgone interest in capital in the form of trees, the monetary value
 210 of the lost trees due to natural mortality and the marginal maintenance cost plus the sum of the monetary

211 values of the direct and indirect changes in growth and mortality of the standing trees. The last neces-
 212 sary condition is just a restatement of the law of motion, and therefore, it will not be discussed here. Fi-
 213 nally, following Sage (1968) the following necessary transversality conditions have to be taken into account.

$$\lambda_0(t) = \lambda(t, l_0), \quad (10)$$

214
$$\lambda(t_1, l) = \frac{dS^{t_1}(x(t_1, l))}{dx}, \quad (11)$$

$$g(E(t, l_m), l_m)\lambda(t, l_m) = \frac{dS^{l_m}(x(t, l_m))}{dx}. \quad (12)$$

215 The first transversality condition, equation (10), requires at every moment that the shadow cost for nursing
 216 trees has to equal the shadow price of the stock evaluated at the diameter l_0 . This transversality condition
 217 is a result of the link between the distributed and the boundary control formed by their common stock
 218 variable. The transversality condition (11) states that the shadow price of the trees has to equal the value
 219 of an additional standing “tree” at the end of the planning horizon. Finally, transversality condition (12)
 220 yields that the shadow price of the trees has to be equal to the value of an additional standing tree with the
 221 maximum diameter.

222 **3 Considerations in the long-run**

223 In this section we analyze to what extent the first order conditions (5) – (9) relate to the first order conditions
 224 of an even-aged forest, commonly expressed by the Faustmann formula. Moreover, we describe the optimal
 225 size distribution of the forest in the steady state, i.e. the steady state distribution of the forest.

226 **3.1 Comparison with the Faustmann formula**

227 From equation (5) we know for an interior solution that $e^{-rt}B_u(u, x) = \lambda(t, l)$, i.e., at every moment of
 228 time it is optimal to cut the number of trees in such a way that the discounted marginal net benefits of timber

229 are equal to the *in situ* value of the standing trees. Hence, we can rewrite equation (8) to obtain a generalized
 230 Faustmann equation.

231 **Proposition 1 (Generalized Faustmann Equation)**

232 *The change in the in situ value of the stock density*

$$\begin{aligned}
 233 \quad & \frac{\partial \lambda(t, l)}{\partial t} + \frac{\partial \lambda(t, l)}{\partial l} g(E(t), l) + B_x - \delta(E(t), l) \lambda(t, l) + \\
 234 \quad & \frac{\pi}{4} l^2 \int_{l_0}^{l_m} \lambda(t, l) \left(\frac{\partial (g_E x(t, l))}{\partial l} + \delta_E x(t, l) \right) dl = r \lambda(t, l). \quad (13)
 \end{aligned}$$

235 *can be interpreted as a generalization of the Faustmann formula.*

236 **Demonstration:** The right-hand side of equation (13) reflects the interest forgone on the capital tied up
 237 in the standing trees. The first two terms of the left-hand side reflect the change in the *in situ* value of
 238 the timber growing for an additional period of time and an increase in diameter by one unit. Given that
 239 $B_x < 0$, the marginal maintenance cost for the stock not being cut down and the monetary loss due to
 240 natural mortality must be subtracted. Moreover, the monetary value of the direct and indirect changes of
 241 growth and mortality processes resulting from a change in the density have to be taken into account. Since
 242 these density dependent changes in growth and mortality are multiplied by the *in situ* value, the last term of
 243 the left-hand side provides a monetary value for these changes. The right-hand side and left-hand side have
 244 to be identical along the optimal path. To see that equation (13) reflects the Faustmann formula as a special
 245 case let us restate the Faustmann formula given by

$$246 \quad P_w f'(T) = r P_w f(T) + \frac{r(P_w f(T) - c e^{rT})}{e^{rT} - 1}, \quad (14)$$

247 where T indicates the age when the entire even-aged stand is cut, P_w is the market price of the wood, $f(T)$
 248 the merchantable volume of wood that a stand of age T produces. The parameter c presents transaction
 249 (logging, processing, transport) and nursing costs.

250 The left-hand sides of the Faustmann formula and of equation (13) reflect the change in the *in situ* value of
251 the forest over time. However, since equation (14) has to be interpreted differently from equation (13), we
252 demonstrate in the appendix that in the case of an even-aged stand, the change in the *in situ* value over time
253 - the equivalent of equation (13) - provides the Faustmann Formula. However, the traditional formula does
254 not take into account the changes in maintenance costs and monetary values from changes in growth and
255 mortality as a result of a change in the diameter of the standing trees. The term $P_w f(T) - c e^{rT}$ reflects the
256 net value of the standing trees in the marketplace and corresponds to $\lambda(t = 0, l)$ if t_1 is identical to T . As t_1
257 increases, future rotations of the forest stand are incorporated into the value of $\lambda(t, l)$ as it denotes the *in situ*
258 value from t to t_1 . Specifically, it holds as t_1 tends to infinity and we obtain that the value of $\lambda(t, l)$ is given
259 by $P_w f(T) - c e^{rT} + \frac{P_w f(T) - c e^{rT}}{e^{rt} - 1}$, where the second term reflects the opportunity cost of the land for
260 an infinite stand rotation. Hence, the change in the *in situ* value $d\lambda/dT$ is given by $P_w f'(T) - r c e^{rT}$ and
261 reflects the left-hand side of equation (14), and $r\lambda = r(P_w f(T) - c e^{rT} + \frac{P_w f(T) - c e^{rT}}{e^{rT} - 1})$ the right-hand
262 side of equation (14) taking into account that the term $r c e^{rT}$ cancels out on both sides, i. e.,

$$263 \quad P_w f'(T) = rP_w f(T) + \frac{r(P_w f(T) - c e^{rT})}{e^{rT} - 1} = r\lambda. \quad \blacksquare \quad (15)$$

264 Thus, based on the concept of the *in situ* value, equation (13) can be interpreted as a generalization of the
265 traditional Faustmann formula, where the forest is distributed over the diameter of the trees. The interpre-
266 tation of equation (13) collapses to the traditional Faustmann formula if l and E are considered constants,
267 i.e., all derivatives with respect to l and E are zero.

268 In practical terms, it is not possible to determine whether the optimal rotation age for most trees is lower
269 or higher under a selective cutting regime than under a clear cutting regime. The precise determination of
270 the optimal rotation age for the trees under a selective cutting regime depends, in particular, on the density
271 effect, on the development of the size of the trees over time, and on the price of timber as a function of the
272 size of the tree.

273 **3.2 Implications of a Steady state on the optimal long-run distribution.**

274 The first order conditions of problem (D) do not permit an analytical solution. To take the analysis further
 275 we consider the case where the forest has reached the steady state, i. e. $\partial\lambda/\partial t = 0$ and $\partial x/\partial t = 0$.

276 The optimal steady state problem is a lumped optimal control problem defined over diameter $l \in [l_0, l_m]$
 277 where the density of trees with diameter l is the control variable. This problem can be thought of as a
 278 situation where the private owner chooses the optimal diameter distribution of the trees in the steady state
 279 resulting from an exogenous shock. Thus, the optimal path of the stock variable $x(l)$ traces out the optimal
 280 steady state distribution. The assumption of an steady-state distribution implies that $E(t)$ is constant, i.e.
 281 the density effect is constant and $E(t) = E$. In this way, the integrodifferential equations (8) and (9) are
 282 ordinary differential equations and are mathematically tractable.

283 By suppressing t , equation (9) can be written as:

$$284 \quad g(E, l) \frac{dx}{dl} = - \left(\frac{dg(E, l)}{dl} + \delta(l) \right) x(l) - u(l), \quad x(0, l) = x_0(l) \quad (16)$$

285 In the case where the benefit function is linear in $u(l)$, the system leads to a corner solution, that is, there
 286 exists a $\tilde{l} \in (l_0, l_m)$ where $u(\tilde{l}) = x(\tilde{l})$, and $x(l) = 0, \forall l > \tilde{l}$. Thus, equation (9), for $l < \tilde{l}$ can be written as

$$287 \quad \frac{dx}{dl} = \frac{- \left(\frac{dg(E, l)}{dl} + \delta(E, l) \right) x(l)}{g(E, l)}.$$

288 Solving this equation yields

$$289 \quad x(l) = \frac{p(l_0)}{g(E, l)} \exp \int_{l_0}^l - \frac{dg(E, s)/ds + \delta(s)}{g(E, s)} ds, \quad (17)$$

290 where we made use of the boundary condition $g(E, l_0)x(l_0) = p(l_0)$. Thus, the optimal long-run distribution

291 will be increasing (decreasing) over the diameter l if the absolute value of $\frac{dg(E, l)}{dl}$ is greater (lower) than
292 $\delta(l)$.

293 For the case of a low mortality rate and fast growing trees, the optimal steady state distribution is increasing
294 in diameter, i.e., the proportion of large trees is relatively high. On the contrary, when the mortality is high
295 and the growth rate is low, the optimal steady state distribution is decreasing in diameter, i.e., the proportion
296 of large trees is relatively low.

297 **4 Empirical study**

298 In practice the necessary conditions (three equations and a system of partial integrodifferential equations)
299 can only be solved analytically under severe restrictions with respect to the specification of the mathematical
300 problem (Muzicant, 1980). Thus, one has to resort to numerical techniques to solve the distributed control
301 problem. Available techniques such as the method of finite differences, the Galerkin method or that of fi-
302 nite elements may be appropriate choices (Calvo and Goetz, 2001). However, all of these methods require
303 the programming of algorithms that are mostly unknown to economists. Therefore we propose a different
304 method called the Escalator Boxcar Train (de Roos, 1988) to describe the evolution of physiologically-
305 structured populations. De Roos (1988) has shown that this technique is an efficient integration technique
306 for structured population models. More details about this technique and how this approach can be extended
307 to account for the incorporation of decision variables can be found in Goetz et al. (2008). The Escalator
308 Boxcar Train (EBT) is based on converting the structuring variable into a state variable of the system by
309 transforming the partial integrodifferential equation into ordinary differential equations over time. More-
310 over, EBT allows the density effect of the biological processes to be considered. In contrast to the other
311 available methods, it can be implemented with standard computer software used to solve mathematical pro-
312 gramming problems.

313 The purpose of the empirical analysis is to determine the optimal selective-logging regime of a diameter-
 314 distributed forest, i.e., the selective logging regime that maximizes the discounted private net benefits from
 315 timber production of a stand of *Pinus sylvestris* (Scots pine) over a time horizon of 300 years.⁷ In this way
 316 it is possible to provide guidance for forest practitioners with respect to the optimal logging pattern, and the
 317 optimal long-run diameter distribution of the stand.

318 4.1 Data and specification of functions

319 In order to solve the decision problem (D) it needs to be transformed into a problem which can be solved
 320 numerically. For this purpose we define $i = 0, \dots, n$ cohorts over the diameter l , i.e., the trees whose
 321 diameters fall within the limits l_i and l_{i+1} are grouped in the cohort i . Hence we can define $X_i(t)$ as the
 322 number of trees, $L_i(t)$ as the average diameter, and $U_i(t)$ as the number of cut trees within the cohort i . \bar{X} ,
 323 \bar{L} , and \bar{U} denote the vectors $\bar{X} = (X_0, \dots, X_n)$, $\bar{L} = (L_0, \dots, L_n)$, and $\bar{U} = (U_0, \dots, U_n)$, respectively.
 324 The vector \bar{X}^0 denotes the initial density of each cohort. As demonstrated in Goetz et al. (2008), the
 325 decision problem (D) can be approximated to the decision problem (D') given by

$$\begin{aligned}
 326 \quad \max_{\bar{U}(t), P(t)} \quad & \int_0^{t_1} B(\bar{X}(t), \bar{L}(t), \bar{U}(t)) e^{-rt} dt - \int_0^{t_1} C(P(t)) e^{-rt} dt \\
 & + S^{t_1}(\bar{X}(t_1), \bar{L}(t_1)) e^{-rt_1} + \int_0^{t_1} S^l(X_n(t), L_n(t)) e^{-rt} dt,
 \end{aligned} \tag{D'}$$

327 subject to the constraints

⁷In countries at higher latitudes the species *Pinus sylvestris* is often considered as shade intolerant, and consequently not suitable for a selective logging regime. However, in countries at mid-range latitudes like Spain natural reproduction requires that older trees protect young trees against heat and water stress in the summer (personal communication by C. Gracia, University of Barcelona, Department of Ecology and CREAF, the Centre for Ecological Research and Forestry Applications). This finding is supported by field experiments reported by Clapham et al. (2002) and Sanchez-Gomez et al. (2006).

$$\frac{dX_i(t)}{dt} = -\delta(\tilde{E}(t), L_i)X_i(t) - U_i(t), \quad \frac{dL_i(t)}{dt} = g(\tilde{E}(t), L_i)$$

$$\frac{dX_0}{dt} = -\delta(\tilde{E}(t), l_0)X_0(t) - \frac{d}{dl}\delta(\tilde{E}(t), l_0)L_0(t) + P(t)$$

$$\frac{dL_0}{dt} = g(\tilde{E}(t), l_0)X_0(t) + \frac{d}{dl}g(\tilde{E}(t), l_0)L_0(t) - \delta(\tilde{E}(t), l_0)L_0(t)$$

$$X_i(0) = \bar{X}^0, \quad g(\tilde{E}(t), l_0)x(t, l_0) = P(t), \quad U_i(t), P(t) \geq 0 \quad U_i(t) \leq X_i(t),$$

where $p(t, l_0)$ is now written as $P(t)$ to unify the notation. The term environment $\tilde{E}(t)$ is determined by

$$\tilde{E}(t) = \sum_{i=0}^n \pi \left(\frac{L_i}{2} \right)^2 X_i(t). \quad (18)$$

For given specifications of the economic and biophysical functions of the model, and a given initial diameter distribution of the trees, \bar{X}^0 , a numerical solution of the decision problem (D') can be obtained. To analyze the sensitivity of the optimal logging regime with respect to a change in the initial diameter distribution, we consider different initial diameter distributions. They were specified as a transformed beta density function $\theta(l)$ with shape parameters γ and ϕ (Mendenhall et al., 1990). The initial forest consists of a population of trees with diameters within the interval $0 \text{ cm} \leq l \leq 50 \text{ cm}$. The distribution of the diameter of the trees is given by:

$$\theta(l; \gamma, \phi) = \begin{cases} \frac{1}{50} \frac{\Gamma(\gamma + \phi)}{\Gamma(\gamma)\Gamma(\phi)} \left(\frac{l}{50} \right)^{\gamma-1} \left(1 - \frac{l}{50} \right)^{\phi-1}, & \gamma, \phi > 0; 0 \leq l \leq 50, \\ 0, & \text{elsewhere,} \end{cases} \quad (19)$$

where $\theta(l; \gamma, \phi)$ denotes the density function of the diameter of trees. Thus, the integral $\int_{l_i}^{l_{i+1}} \theta(l; \gamma, \phi) dl$ gives the proportion of trees lying within the range $[l_i, l_{i+1})$. The beta density function is used because it is defined over a closed interval and allows a great variety of distinct shapes of the initial diameter distributions of the trees to be defined. We defined $l_0 = 0$ and $l_m = 80$. Within this interval we concentrate on the interval

343 [0, 50] and divide it into 10 initial subintervals of identical length. In this way, each cohort comprises trees
 344 that differ in their diameter by a maximum of 5 cm, and can therefore be considered as homogeneous. The
 345 initial number of trees in each cohort, $X_i(0)$, $i = 1, \dots, n$, is calculated in such a way that the basal area
 346 of the stand is constant (25 m²/ha) in order to allow for comparisons between the results of the different
 347 optimization outcomes.

348 The function $B(\bar{X}(t), \bar{L}(t), \bar{U}(t))$ accounts for the net benefits of the timber at time t and is defined as:
 349 $[\sum_{i=0}^n (\rho(L_i(t)) - vc) tv(L_i(t)) mv(L_i(t)) U_i(t)] - [mc(X(t))]$, where $X(t) = \sum_{i=0}^n X_i(t)$. The first
 350 term in square brackets denotes the sum of the revenue of the timber sale minus the cutting cost of each
 351 cohort i , and the second term, $mc(X(t))$, accounts for the maintenance cost. The parameter $\rho(L_i)$ denotes
 352 the timber price per cubic meter of wood as a function of the diameter, $tv(L_i)$ is the total volume of a tree
 353 as a function of its diameter, $mv(L_i)$ is the part of the total volume of the tree that is marketable and vc is
 354 the variable cutting cost.

355 Timber price per cubic meter was taken from a study by Palahí and Pukkala (2003), who analyzed the
 356 optimal management of a *Pinus sylvestris* forest in a clear-cutting regime. They estimated a polynomial
 357 function given by $\rho(L) = \text{Min}[-23.24 + 13.63\sqrt{L}, 86.65]$, which is an increasing and strictly convex
 358 function, for a diameter lower than 65cm. At $L = 65$ the price reaches its maximum value, thus, it is
 359 considered constant for $L > 65$. Data about costs were provided by the consulting firm Tecnosylva, which
 360 elaborates forest management plans throughout Spain. The logging cost comprises logging, delimiting,
 361 and collecting and removing the residues, and it is given by $vc = 15$ euros per cubic meter of logged
 362 timber. The maintenance cost is an increasing function of the number of stems per hectare, and is given by
 363 $mc(X) = 10 + 0.0159X + 0.0000186X^2$. The nursing cost is linear in the amount of ingrowing trees and
 364 is given by $C(P) = 0.73P$.

365 The value of the parameters of tree volume, $tv(L_i)$, and the marketable part of the tree volume, $mv(L_i)$,
 366 are estimated using information provided by a study by Cañellas et al. (2000). The tree volume follows

367 the allometric relation $tv(L) = 0.00157387L^{1.745087}$, and the marketable part of the volume of timber of
368 each tree is an increasing function of the diameter, given by $mv(L) = 0.699 + 0.0004311L$. The thinning
369 and nursing period, Δt , is set equal to 10 years, which is a common practice for a *Pinus sylvestris* forest
370 (Cañellas et al., 2000).

371 To determine the dynamics of the forest the growth of a diameter-distributed stand of *Pinus sylvestris* with-
372 out thinning was simulated with the bio-physical simulation model GOTILWA (Growth Of Trees Is Limited
373 by Water⁸). About 100 different simulation runs were conducted by varying the initial diameter distribution.
374 The results of the simulation were used to estimate the function $g(E, L_i)$, which describes the rate of diame-
375 ter change. It was specified as a von Bertalanffy growth curve (von Bertalanffy, 1957), generalized by Millar
376 and Myers (1990) which allows the rate of growth of the diameter to vary with environmental conditions.
377 Thus, the function $g(E, L_i) = (l_m - L_i)(\beta_0 - \beta_1 BA)$ was estimated by the method of *OLS*, where β_0 and
378 β_1 are proportionality constants, and BA is the sum of the basal area of all trees. The estimation yielded
379 the growth function: $g(E, L_i) = (80 - L_i)(0.0070177 - 0.000043079 BA)$. Other functional forms of
380 $g(E, L_i)$ were evaluated as well, but they explained the observed variables to a lesser degree.⁹

381 As GOTILWA only allows the survival or death of an entire cohort to be simulated but not the survival or
382 death of an individual tree, it was not possible to obtain an adequate estimation of the function $\delta(E, L_i)$
383 describing the mortality of the forest. Nevertheless, the information provided by Tecnosylva suggests that
384 in a managed forest, the mortality rate can be considered almost constant over time and independent of the
385 diameter. Thus, according to the data supplied by Tecnosylva, $\delta(E, L_i)$ was chosen to be constant over time

⁸This program has been developed by C. Gracia and S. Sabaté, University of Barcelona, Department of Ecology and CREAM (Centre de Recerca Ecològica i Aplicacions Forestals), Autonomous University of Barcelona, respectively. It simulates growth and mortality and allows one to explore how the life cycle of an individual tree is influenced by the climate, the characteristics of the tree itself and environmental conditions given by the total basal area of the trees. The model is defined by 11 input files specifying more than 90 parameters related to site, soil composition, tree species, photosynthesis, stomatal conductance, forest composition, canopy hydrology, and climate.

⁹The discretization scheme employed has the advantage that the set of ordinary difference equations is derived from the underlying partial integro-differential equation, whereas in previous literature, for example (Haight et al., 1985), the set of ordinary difference equations was set up ad hoc. Moreover, the EBT method requires only the differential equation $dl/dt = g(\cdot)$ to be estimated, whereas the approach followed in the previous literature would require a complex system of difference equations to be estimated simultaneously.

386 and equal to 0.01 for each cohort.

387 **4.2 Optimization results**

388 The mathematical optimization problem (D') was programmed in GAMS (General Algebraic Modeling Sys-
389 tem) (Brooke et al., 1992). For the numerical solution of this problem the Conopt3 solver, available within
390 GAMS, was employed. For a given initial distribution, the numerical solution of the problem determines
391 for every 10-year period the optimal logging, U_i , and planting, P ; the optimal values of the state variables,
392 X_i and L_i ; and consequently, economic variables, such as the revenue from timber sales and the cutting,
393 planting, and maintenance costs. Optimizations with different random initializations of the control variables
394 were carried out to assure that the numerical method provides solutions that are independent of the initially
395 chosen values for the numerical optimization technique. All optimizations were carried out on a per-hectare
396 basis.

397 **Selective Logging Regime**

398 Forest managers who want to maximize net timber benefits have to decide on the intensity of cutting, that
399 is, how many trees of diameter L_i have to be cut at each moment of time. To calculate the optimal logging
400 regime we assume that the initial diameter distribution of the trees is given by a beta density function with
401 parameters $\gamma = 0.8$ and $\phi = 0.2$, corresponding to a young forest distribution. Table 1 summarizes the
402 results of the optimization where a discount rate of 2% was assumed. It can be observed that all economic
403 and biophysical variables show a cyclical pattern over time in which the phase of the pattern is maintained
404 over time, but the amplitude decreases. In the long-run, the forest consists of approximately 927 trees, and
405 approximately 123 of these trees are logged each 10-year period. The volume of the logged trees is 73.64
406 m^3 , of which 52.43 m^3 is marketable timber. The current-value revenue from the sale of this amount of
407 timber minus the logging cost is approximately 1900 euros per hectare. The current-value net benefits of the
408 forest in the long run are nearly 1000 euros. Following this regime, the total sum of discounted net benefits

409 of the forest over 300 years is 7794.27 euros per hectare.

410

Table 1

411 To illustrate the optimal evolution of the forest, Figure 1 a) to f) depicts the change in the histogram of
412 the forest distribution over the planning horizon. It must be noted that the width of the diameter intervals
413 is only homogeneous at the initial point in time, but the width of the different cohorts changes with the
414 evolution of the forest over time. Thus, the size of the bar of a particular cohort is chosen so that its area
415 corresponds to the number of trees in that cohort. The slightly shaded bars indicate the number of trees that
416 should be logged within each 10-year period, while the heavily shaded bars stand for the number of trees
417 that should remain in the stand. Figure 1 shows that the initially skewed diameter distribution is gradually
418 flattened over time. However, during this process the distribution varies in the form of a wave leading to an
419 almost u-shaped distribution after 50 years. Figure 1 also shows that it takes more than 100 years to reach a
420 diameter distribution of the trees which is relatively stable.¹⁰

421

Figure 1

422 **Effects of a Change in the Initial Diameter Distribution on the Optimal Selective-Logging Regime**

423 To illustrate how the initial diameter distribution of the trees alters the optimal selective logging regime,
424 problem (D') has also been solved for an old forest distribution ($\gamma = 2, \phi = 0.8$), a bell shaped distribution
425 ($\gamma = \phi = 2$), a U-shaped distribution ($\gamma = \phi = 0.5$), a uniform distribution ($\gamma = \phi = 1$) and a non-
426 structured forest (even-sized, $\gamma = \phi = \infty$). Figure 2 depicts the optimal evolution of the weighted average
427 (2a) and standard deviation (2b) of the diameter distribution over time for the analyzed initial distributions.¹¹

¹⁰Please note that the average diameter of the bars (cohorts) is not constant over time. This is explained by the fact that the trees of a cohort always stay together and do not move from one cohort to another. However, since the trees grow the average diameter of the cohort increases as the cohort moves along the time axis.

¹¹Figure 2 shows the development of the average and standard deviation of the diameter distribution of three different initial diameter distributions. The remaining three initial diameter distributions are not depicted because they follow the same pattern, and their graphical representation would obstruct the interpretation of Figure 2.

428 It shows that the average diameter of the different distributions tends to converge after approximately 200
429 years, as the amplitude and phase of the cyclical behavior vanishes. Additionally, Figure 2 shows that the
430 standard deviation of the distributions is governed by the same cyclical pattern. When the mean diameter
431 and standard deviation of the initial diameter distribution are close to that of the steady state (young forest
432 with $\gamma = 0.8, \phi = 2$), the cyclical evolution of these variables is less pronounced, implying that the benefits
433 will be more stable over time. In general, it can be observed that the long-run mean and standard deviation
434 tend to the values of 17 and 9, respectively, for all considered parameter values of γ and ϕ of the diameter
435 distribution. Hence, the steady-state distribution of the diameter of the trees is independent of the initial
436 distribution of the trees. This result confirms the supremacy of the normal forest as the optimal forest
437 management objective.

438 Figure 2

439 Moreover, we conducted a sensitivity analysis to determine the effect of a change in the initial basal area on
440 the steady-state distribution. Figure 3 illustrates the optimal evolution of the weighted average diameter (3a)
441 and the standard deviation (3b) of the diameter over time for a young forest, given the initial basal areas of
442 15, 25 and 35 m²/ha. One can see from Figure 3 that the long-run mean and standard deviation tend to the
443 same values as in Figure 2 (17 and 9, respectively). This result shows that the steady-state distribution is not
444 only independent from the initial diameter distribution but is also independent from the initial basal area.

445 Figure 3

446 We also conducted a sensitivity analysis to evaluate how the optimal management regime of a forest changes
447 as a result of a variation in the discount rate. Thus, we solved problem (D') for a young forest distribution,
448 given discount rates of 3% and 4%. Figure 4 depicts the optimal evolution of the mean diameter (4a) and
449 standard deviation (4b) over time resulting from the optimizations. Figure 4 shows that the discount rate has
450 a significant influence on the optimal selective logging regime. An increase in the discount rate produces,

451 in the long run, a decrease in the average diameter at which the trees are cut, that is, the trees are cut earlier.
452 However, it can be observed in Figure 4b that the initial diameter distributions stabilize in the long run,
453 independently of the chosen discount rate, i.e., the steady state distribution is independent of the discount
454 rate.

455 Figure 3

456 At the end of section 3.2 we stated that the shape of the distribution at the steady state depends on growth
457 and mortality rates. To illustrate this point, the optimal selective logging regime in the case of an initial
458 diameter distribution of trees corresponding to a young forest is calculated for a higher mortality rate of
459 $\delta = 0.1$ compared to the previously chosen mortality rate of 0.01. The histogram of the resulting long-run
460 distribution is depicted in Figure 5. It shows that an increase in the mortality rate causes the optimal steady-
461 state distribution to be decreasing in diameter.

462 Figure 5

463 Our results show that in the presence of density effects the steady state distribution tends to a normal forest
464 and is independent of the initial distribution of trees. For a comparison of our results with the previous
465 results in the literature we refer primarily to the article by Salo and Tahvonen (2002) since the employed
466 model is still the up-to-date cornerstone for other works. In the results of their work the optimal long-run age
467 distribution is non cyclical only when the length of the discretely measured time period converges towards
468 zero, when the discount rate is zero or when the Faustmann rotation is not unique. Under these conditions
469 a normal forest may result. Otherwise many other outcomes, which do not correspond to a normal forest
470 are possible. Salo and Tahvonen (2002) observe that the analytical framework of a time discrete model is
471 the principal cause for the emergence of logging cycles. The model presented in this article is based on
472 a different analytical framework, which allows relaxing assumptions made by Salo and Tahvonen (2002):
473 the stand is structured with respect to diameter, the economically more relevant variable, and not age; the

474 cleared land does not have to be replanted immediately; the forest manager can vary the number of planted
475 trees per hectare; and it allows partial harvesting of a cohort. Uusivuori and Kuuluvainen (2005) used an
476 analytical framework very similar to Salo and Tahvonen (2002) but allowed also for partial harvesting of
477 the different age classes, leading to long-run distributions being either cyclical or not. According to their
478 findings the noncyclical forest is typically not a normal forest. However, the authors assume in contrast,
479 to this study, and to the article by Salo and Tahvonen (2002), that the price for one cubic feet of timber is
480 independent of the age of the trees. As mentioned above the price of timber per cubic feet usually increases
481 with the diameter (age) of the trees since the wood can be used for the production of more valuable goods.
482 One would expect this assumption to affect the optimal logging regime. For this reason, and due to the fact
483 that trees in the model of Uusivuori and Kuuluvainen (2005) do not only provide timber but also amenity
484 values, it is not possible to compare their results with the results of the cited literature.

485 **5 Conclusions**

486 This article presents a theoretical model that allows us to determine the optimal management of a diameter-
487 distributed forest where the growth process of the trees depends not only on their individual sizes but also
488 on the size distribution of the trees within the entire stand. This modeling framework allows us to take into
489 account the fact that the life cycle of each individual tree is affected by the other trees since they compete
490 for light, nutrients, and space. The density dependent formulation of the biological growth process leads
491 to a partial integrodifferential equation. To determine the optimal forest management, the corresponding
492 economic decision problem can be formulated as a distributed optimal control problem where the control
493 variables and the state variable depend on both time and the diameter of the trees.

494 The resulting necessary conditions of this problem allow the derivation of an analytical expression which
495 can be interpreted as a generalization of the Faustmann formula. Since the necessary conditions of this
496 problem include a system of partial integrodifferential equations, it usually cannot be solved analytically.

497 Thus, the Escalator Boxcar Train method is proposed to solve the problem numerically. The method allows
498 the partial integrodifferential equation to be transformed into a set of ordinary differential equations and
499 thereby approximate the distributed optimal control problem by a standard optimal control problem. In con-
500 trast to the existing literature, the resulting optimization problem can be solved numerically using standard
501 mathematical programming techniques and does not require programming complex numerical algorithms.

502 To determine the optimal selective logging regime of a diameter-distributed and privately owned forest
503 where individual trees compete for scarce resources, an empirical analysis is conducted. It shows that the
504 long-run mean and variance of the diameter distribution for the different types of analyzed forests consid-
505 ered tend to a common value, giving rise to a normal forest. Although the diameter distribution of the forests
506 in the steady-state is independent of the initial distribution, the competition between individuals belonging
507 to the same population affects the transition paths to the steady-state distribution, and therefore the opti-
508 mal selective-logging regimes in the first periods differ considerably depending on the distribution of the
509 individual characteristics over the entire population.

Appendix

510

511 In the case of an even-aged stand decision problem and an infinite planning horizon, problem (D) can be
 512 simplified, and is given by

$$513 \quad \max_{u(t)} \int_0^{\infty} \tilde{B}(x(t), u(t)) e^{-rt} dt, \quad (\text{DE})$$

514 subject to the constraints

$$515 \quad \dot{x}(t) = \tilde{g}(x(t)) - u(t), \quad x(0) = x_0, \quad u(t) \geq 0.$$

516 The state and control variables do not depend anymore on the diameter l and since all trees are planted at the
 517 same time, the planting costs can be incorporated into the net benefit function of the timber, $\tilde{B}(x(t), u(t))$.
 518 Moreover, the residual value of the stand approaches zero as t approaches infinity, and $\tilde{g}(x(t))$ denotes
 519 the growth of the biomass. With these provisions the current value Hamiltonian, \mathcal{H}^{DE} , yields $\mathcal{H}^{DE} =$
 520 $\tilde{B}(x, u) + \lambda(\tilde{g}(x) - u)$ and the first order conditions are given by

$$521 \quad \mathcal{H}_u^{DE} = \tilde{B}_u - \lambda + \mu_1 = 0, \quad (\text{A. 1})$$

$$522 \quad \dot{\lambda} = r\lambda - \tilde{B}_x, \quad \lim_{t \rightarrow \infty} \lambda(t) = 0 \quad (\text{A. 2})$$

$$523 \quad \dot{x} = \tilde{g}(x) - u, \quad x(0) = x_0. \quad (\text{A. 3})$$

524 Unfortunately, this formulation does not allow the Faustmann formula to be derived. Yet, a redefinition of
 525 the variables yields an equivalent model that in turn provides the Faustmann formula. Let

$$526 \quad x(t) = T(t), \text{ and therefore } \dot{x} = \dot{T},$$

$$527 \quad \lambda(t) = \varphi(t), \text{ where } \varphi \text{ is the new costate variable,}$$

$$528 \quad u(t) = 1 - I(t), \text{ where } 0 \leq I \leq 1 \text{ indicates the share of trees that is invested, i.e. not cut,}$$

$$529 \quad \dot{x} = \tilde{g}(x) - u = 1 - (1 - I) = I, \text{ where } \tilde{g}(x) \equiv 1, \text{ and given the previous definitions we obtain } \dot{T} = I,$$

530 $\tilde{B}(x, u) = rF(t)(1 - I)$, where $F(T) = P_w f(T) - c e^{rT} + v$. The parameter v denotes the land expectation
 531 or site value.

532 Thus, the new decision problem formulated originally by Weitzman (2003) yields

$$533 \quad \max_{I(t)} \int_0^{\infty} r(F(T(t))(1 - I(t))e^{-rt} dt, \quad (\text{DF})$$

534 subject to the constraints

$$535 \quad \dot{T}(t) = I(t), \quad T(0) = 0, \quad 0 \leq I(t) \leq 1.$$

536 The current value Hamiltonian, \mathcal{H}^{DF} , yields $\mathcal{H}^{DF} = rF(T)(1 - I) + \varphi I$ and the first order conditions
537 result in

$$538 \quad \mathcal{H}_I^{DF} = rF(T) - \varphi + \mu_2 = 0, \quad (\text{A. 4})$$

$$539 \quad \dot{\varphi} = r\varphi - rF'(T)(1 - I), \quad (\text{A. 5})$$

$$540 \quad \dot{T} = I. \quad (\text{A. 6})$$

541 Utilizing the substitution we made for the formulation of the decision problem (DE) in the first order
542 conditions of problem (DE) shows that equations (A. 1) – (A. 3) are identical to the first order conditions
543 (A. 4) – (A. 6) of problem (DF). Hence, their solutions are identical. Utilizing equation (A. 4) we see that
544 $\varphi = rF(T) + \mu_2$. Hence, $\dot{\varphi} = rF'(T)\dot{T} + \mu_2$. Substituting this expression in equation (A. 5) and using
545 equation (A. 6) yields

$$546 \quad rF'(T)I + \dot{\mu}_2 = r^2F(T) + \mu_2 - rF'(T)(1 - I). \quad (\text{A. 7})$$

547 Thus,

$$548 \quad \dot{\mu}_2 = r^2F(T) + \mu_2 - rF'(T). \quad (\text{A. 8})$$

549

550 As long as $I > 0$, i.e., we do not cut, μ_2 will be zero and we obtain

$$551 \quad r = \frac{F'(T)}{F(T)} = \frac{P_w f'(T) - r c e^{rT}}{P_w f(T) - c e^{rT} + v}. \quad (\text{A. 9})$$

552 As shown by Weitzman (2003), equation (A. 9) is identical to the reformulation of the Faustmann formula
553 given by

554
$$P_w f'(T) = rP_w f(T) + \frac{r(P_w f(T) - c e^{rT})}{e^{rT} - 1}, \quad (\text{A. 10})$$

555 where v has been replaced by the maximum site value $\frac{P_w f(T) - c}{e^{-rT} - 1}$. Hence we can conclude that given the
556 optimal value of I , equation (A. 2), which is equivalent to equation (A. 5), provides the Faustmann formula
557 for an even-aged stand.

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Table 1: Optimal Selective-Logging Regime (where the initial diameter distribution is determined by $\gamma = 0.8$ and $\phi = 2$)

Year	Number of trees ^(a)	Planted trees	Logging					Maintenance cost (Euros/ha)	Planting cost (Euros/ha)	Net benefit (Euros/ha)	Discounted net benefit (Euros/ha)
			Logged trees	BA (m ² /ha)	Volume (m ³ /ha)	Timber (m ³ /ha)	Revenue - logging cost (Euros/ha) ^(b)				
0	820	202	115	12.43	98.78	70.66	3211.29	-355.74	-121.19	2734.36	2734.36
10	947	107	67	5.05	42.19	30.05	1127.83	-417.68	-64.33	645.81	529.79
20	959	119	85	5.89	49.74	35.41	1274.32	-424.12	-71.48	778.71	524.05
30	964	132	105	6.79	57.78	41.11	1430.57	-426.60	-79.13	924.85	510.58
40	958	144	129	7.94	68.06	48.40	1642.03	-423.27	-86.52	1132.23	512.78
50	939	137	154	9.19	79.06	56.20	1874.82	-413.53	-82.11	1379.18	512.40
60	929	133	137	8.16	70.17	49.88	1662.47	-408.81	-79.52	1174.14	357.86
70	935	163	118	8.50	71.39	50.83	1872.41	-411.57	-97.50	1363.34	340.87
80	909	121	179	12.94	108.66	77.37	2850.05	-398.61	-72.45	2378.99	487.95
90	921	129	100	7.18	60.31	42.94	1576.15	-404.48	-77.07	1094.59	184.18
100	929	136	111	7.93	66.69	47.49	1738.22	-408.63	-81.17	1248.42	172.32
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
200	928	136	128	9.08	76.41	54.40	1986.89	-408.15	-81.22	1497.52	28.53
300	927	136	123	8.75	73.64	52.43	1914.72	-407.72	-81.48	1425.52	3.75

^(a) The number of trees in the forest is calculated just after the trees are planted, and before the thinning takes place.

^(b) All monetary values apart from the discounted net benefit in the last column of the table are expressed as current values.

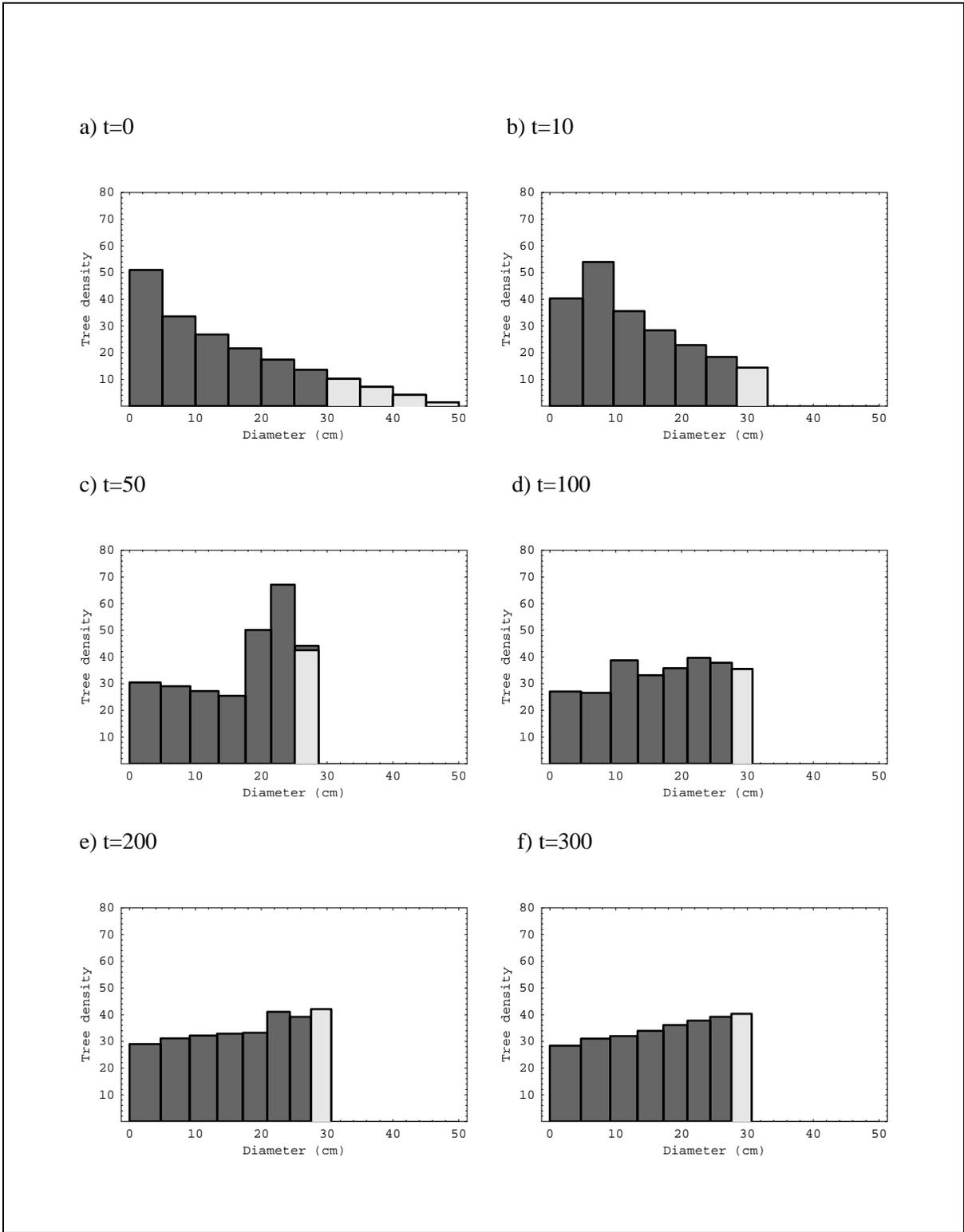


Figure 1: Evolution of the Optimal Diameter Distribution

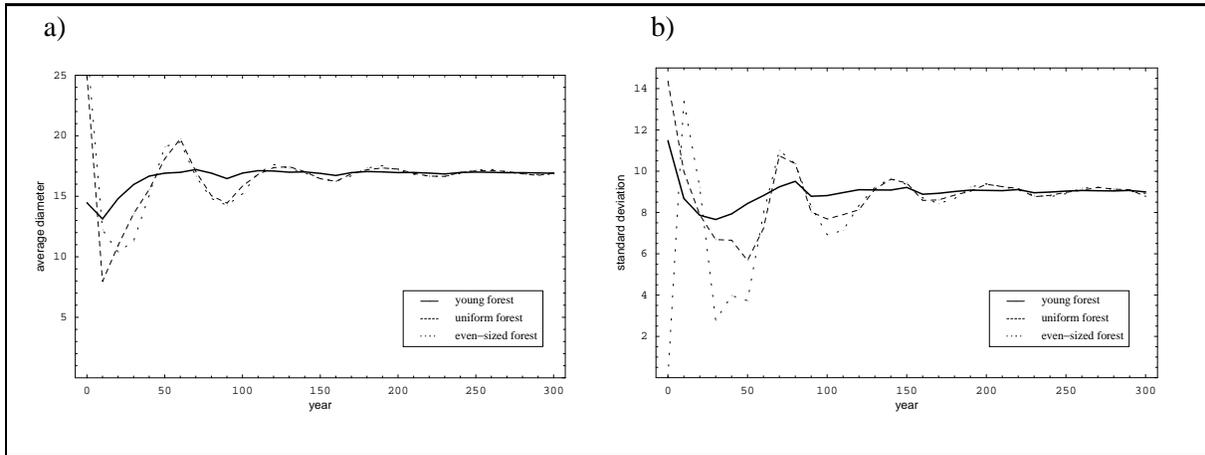


Figure 2: Evolution of the Optimal Diameter Distribution for Different Initial Distributions

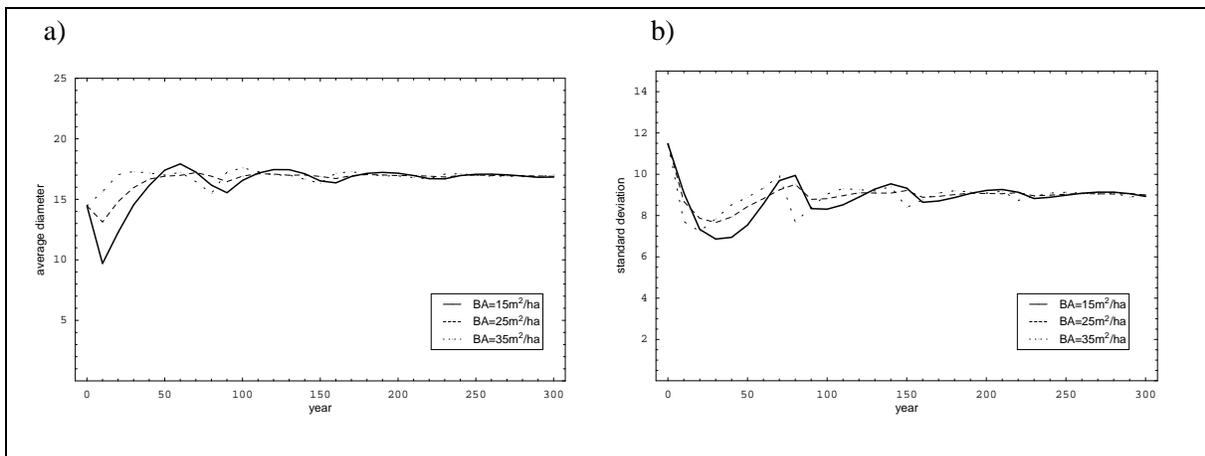


Figure 3: Evolution of the Optimal Diameter Distribution for Different Initial Basal Areas

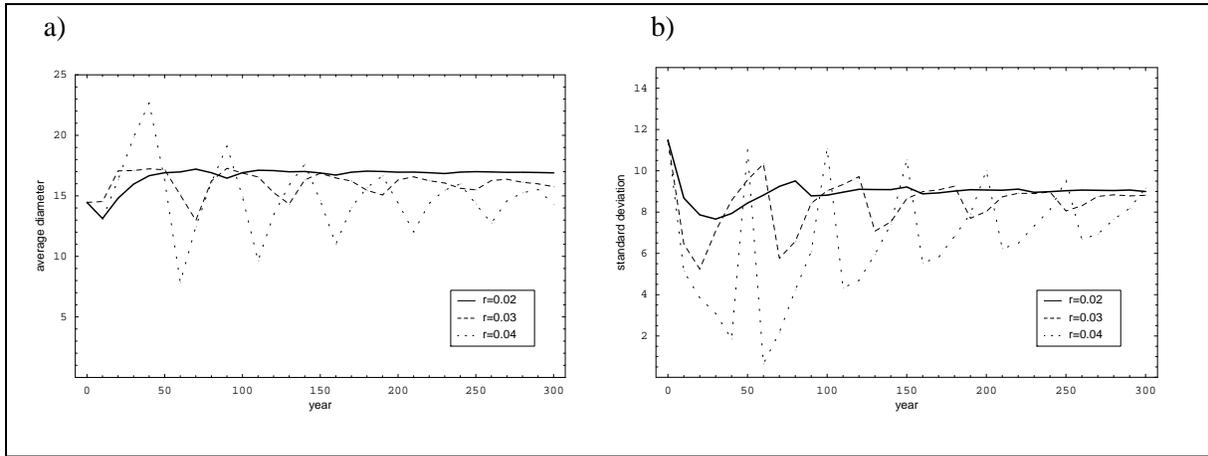


Figure 4: Evolution of the Optimal Diameter Distribution for Different Different Levels of the Discount Rate

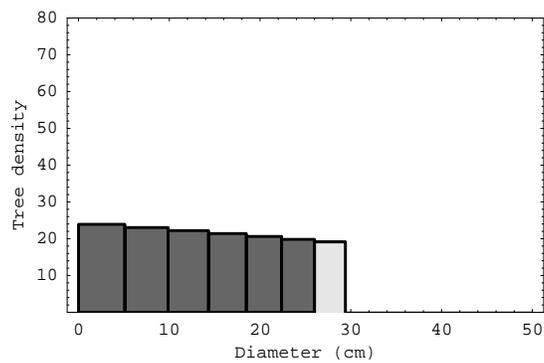


Figure 5: "Long-Run" Diameter Distribution of the Trees Given a Mortality Rate of 0.1