Aitchison Geometry for Probability and Likelihood as a new approach to mathematical statistics

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Abstract

The Aitchison vector space structure for the simplex is generalized to a Hilbert space structure $A^2(P)$ for distributions and likelihoods on arbitrary spaces. Central notations of statistics, such as Information or Likelihood, can be identified in the algebraical structure of $A^2(P)$ and their corresponding notions in compositional data analysis, such as Aitchison distance or centered log ratio transform.

In this way very elaborated aspects of mathematical statistics can be understood easily in the light of a simple vector space structure and of compositional data analysis. E.g. combination of statistical information such as Bayesian updating, combination of likelihood and robust M-estimation functions are simple additions/perturbations in $A^2(P_{prior})$. Weighting observations corresponds to a weighted addition of the corresponding evidence.

Likelihood based statistics for general exponential families turns out to have a particularly easy interpretation in terms of $A^2(P)$. Regular exponential families form finite dimensional linear subspaces of $A^2(P)$ and they correspond to finite dimensional subspaces formed by their posterior in the dual information space $A^2(P_{prior})$.

The Aitchison norm can identified with mean Fisher information. The closing constant itself is identified with a generalization of the cummulant function and shown to be Kullback Leiblers directed information. Fisher information is the local geometry of the manifold induced by the $A^2(P)$ derivative of the Kullback Leibler information and the space $A^2(P)$ can therefore be seen as the tangential geometry of statistical inference at the distribution P.

The discussion of $A^2(P)$ valued random variables, such as estimation functions or likelihoods, give a further interpretation of Fisher information as the expected squared norm of evidence and a scale free understanding of unbiased reasoning.

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